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THE (1, 2) CORRESPONDENCE ASSOCIATED WITH THE CUBIC SPACE INVOLUTION OF ORDER TWO*†

BY

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1. In two previous papers‡ we have developed a process of obtaining (1, 2) correspondences between two spaces from which rational space involutions can be derived. In the present paper we consider the problem of finding the (1, 2) correspondence associated with a given rational space involution and particularly with the non-monoidal cubic involution of order two. The results obtained are of special interest on account of the configuration of the fundamental elements.

2. Consider an involution in a space x such that the pairs of conjugate points are (y), (y'). We are concerned with the birational transformation which interchanges the points of each pair. The lines joining conjugate points (y), (y') constitute either a congruence or a complex. In the former case each line contains an infinite number of pairs; in the latter, a finite number of pairs lies on each line of the complex. In the case of the cubic involution, the point (y') is the point of intersection of the polar planes of the point (y) with respect to three quadrics $A(x) = 0$, $B(x) = 0$, $C(x) = 0$. The quadric of the net that contains the line $(y)(y')$ has the equation

$$\begin{vmatrix} A(x) & B(x) & C(x) \\ A(y) & B(y) & C(y) \\ A(y') & B(y') & C(y') \end{vmatrix} = 0. \tag{1}$$

There is one point (X) on the line $(y)(y')$ such that the tangent plane at this point to the quadric (1) shall pass through P_8, a selected one of the eight basis points of the net of quadrics. Conversely, given a point (X), there is

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‡ *Certain types of involutorial space transformations*, these Transactions, vol. 20 (1919), pp. 185—202 and vol. 21 (1920), pp. 52—78.

one quadric of the net (1) that contains the line $(X)\,P_8$, and on this quadric passes through (X) a unique second line on the quadric. On this line is just one pair of conjugate points in the involution I, namely, the points in which the line is met by its conjugate cubic curve in I. Hence the lines (y) (y') constitute a complex, and this complex is rational; moreover, since there is a $(1, 2)$ correspondence between the points (X) and the points (y), the cubic involution I is also rational.

By choosing four of the eight basis points as the vertices of the tetrahedron of reference we may take for three quadrics of the net

$$\begin{aligned} A(x) &= x_4 x_1 + a_1 x_2 x_3 + a_2 x_3 x_1 + a_3 x_1 x_2 = 0, \\ B(x) &= x_4 x_2 + b_1 x_2 x_3 + b_2 x_3 x_1 + b_3 x_1 x_2 = 0, \\ C(x) &= x_4 x_3 + c_1 x_2 x_3 + c_2 x_3 x_1 + c_3 x_1 x_2 = 0. \end{aligned} \tag{2}$$

Let $P_8 = (0, 0, 0, 1)$. The tangent plane to (1) at P_8 contains (X). If we write

$$X_i = k' y_i + k y_i', \tag{3}$$

we find, by putting (X) for (x) in the equation of the tangent plane, that

$$k = \begin{vmatrix} y_1 & y_2 & y_3 \\ A(y) & B(y) & C(y) \\ A(y') & B(y') & C(y') \end{vmatrix}$$

and

$$-k' = \begin{vmatrix} y_1' & y_2' & y_3' \\ A(y) & B(y) & C(y) \\ A(y') & B(y') & C(y') \end{vmatrix}.$$

The three polar relations $A(y, y') = 0$, $B(y, y') = 0$, $C(y, y') = 0$ determine the values of y_i' as cubic functions of (y). The surfaces $y_i' = 0$ have in common a sextic curve γ_6, locus of the vertices of the quadric cones contained in the net of quadrics. It follows that $k = 0$ contains γ_6 triply, and k' contains γ_6 doubly, and the surfaces $X_i = 0$, of order 12, contain γ_6 triply. Since $k = 0$ is of order 11, and $k' = 0$ is of order 9, it follows further that apart from γ_6, the surfaces (3) have in common a curve of order 63. Hence two surfaces of the system (3) meet in a variable curve of order 27.

For our purpose, it will be convenient to use the following notation. Let a general quadric of the net be expressed in the form

$$\lambda_1 A(x) + \lambda_2 B(x) + \lambda_3 C(x) = 0. \tag{4}$$

It will contain the points (y), (y') when

$$\begin{aligned} &\lambda_1 A(y)+\lambda_2 B(y)+\lambda_3 C(y)=0,\\ &\lambda_1 A(y')+\lambda_2 B(y')+\lambda_3 C(y')=0. \end{aligned} \tag{5}$$

If we now write $p_{ik}=y_i y_k'-y_k y_i'$ we may express equations (3) in the form

$$\begin{aligned} X_1 &= p_{12}\lambda_2+p_{13}\lambda_3,\\ X_2 &= p_{21}\lambda_1+p_{23}\lambda_3,\\ X_3 &= p_{31}\lambda_1+p_{32}\lambda_2,\\ X_4 &= p_{41}\lambda_1+p_{42}\lambda_2+p_{43}\lambda_3. \end{aligned} \tag{6}$$

The surfaces $p_{ik}=0$ are of order four, and pass through γ_6 simply. From (5) we see that the surfaces $\lambda_i=0$ are of order 8 and pass through γ_6 doubly.

3. Fundamental elements. Consider a point $(1, y_2, y_3, y_4)$ near the basis point $P_1=(1, 0, 0, 0)$. From the polar relations $A(y, y')=0$, $B(y, y')=0$, $C(y, y')=0$ it follows that $(y')=(1, -y_2, -y_3, -y_4)$ to the first order of small numbers; hence

$$X_2=2y_2,\quad X_3=2y_3,\quad X_4=2y_4.$$

From the relation $\lambda_1 A(y)+\lambda_2 B(y)+\lambda_3 C(y)=0$ we have

$$\lambda_1(y_4+a_2y_3+a_3y_2)+\lambda_2(b_2y_3+b_3y_2)+\lambda_3(c_2y_3+c_3y_2)=0;$$

hence, from the preceding relations,

$$\lambda_1(X_4+a_2X_3+a_3X_2)+\lambda_2(b_2X_3+b_3X_2)+\lambda_3(c_2X_3+c_3X_2)=0.$$

Moreover, since $\lambda_1 A(x)+\lambda_2 B(x)+\lambda_3 C(x)=0$ contains the line

$$\frac{x_1}{X_1}=\frac{x_2}{X_2}=\frac{x_3}{X_3},$$

we also have

$$\begin{aligned} &\lambda_1 X_1+\lambda_2 X_2+\lambda_3 X_3=0,\\ &\lambda_1(a_1X_2X_3+a_2X_3X_1+a_3X_1X_2)+\lambda_2(b_1X_2X_3+b_2X_3X_1+b_3X_1X_2)\\ &\quad+\lambda_3(c_1X_2X_3+c_2X_3X_1+c_3X_1X_2)=0. \end{aligned}$$

Upon eliminating λ_1, λ_2, λ_3 from these three equations we obtain as image of the basis points $P_1=(1, 0, 0, 0)$ a quartic surface.

Now consider the quartic surface $p_{23} = 0$, that has the equation

$$\begin{vmatrix} x_4 + a_2 x_3 + a_3 x_2 & a_1 x_3 + a_3 x_1 & a_1 x_2 + a_2 x_1 & x_1 \\ b_2 x_3 + b_3 x_2 & x_4 + b_1 x_3 + b_3 x_1 & b_1 x_2 + b_2 x_1 & x_2 \\ c_2 x_3 + c_3 x_2 & c_1 x_3 + c_3 x_1 & x_4 + c_1 x_2 + c_2 x_1 & x_3 \\ 0 & -x_3 & x_2 & 0 \end{vmatrix} = 0.$$

By multiplying the columns by x_1, x_2, x_3, x_4 respectively and adding to the second we obtain

$$\begin{vmatrix} x_4 + a_2 x_3 + a_3 x_2 & A(x) & a_1 x_2 + a_2 x_1 & x_1 \\ b_2 x_3 + b_3 x_2 & B(x) & b_1 x_2 + b_2 x_1 & x_2 \\ c_2 x_3 + c_3 x_2 & C(x) & x_4 + c_1 x_2 + c_2 x_1 & x_3 \\ 0 & 0 & x_2 & 0 \end{vmatrix} = 0.$$

After removing the extraneous factor x_2 the result is exactly of the same form in (x) as the image in (X) of the point P_1 in (y).

The surface $p_{23} = 0$ has $P_8 P_1$ for double line; the lines joining each of these points to the other six basis points lie on the surface.

The correspondence between (y) and (X) is clearly symmetrical in the seven points P_i, hence the image of each of them is a quartic surface. By introducing the notation (81) for the surface having the line $P_8 P_1$ for double line, we can readily write down the complete configuration of the lines on each of the seven surfaces.

4. **The twenty-one basis lines in** (y). The lines joining the seven basis points P_i, $i \neq 8$, to each other belong to every surface of the web of s_{12}. Consider the point $(y_1, y_2, 0, 0)$ on the line $y_3 = 0, y_4 = 0$. The conjugate point (y') in I is $(y_2, -y_1, 0, 0)$, hence the line is invariant in I and no points are fundamental except the two basis points which it contains. Since the line contains an infinite number of pairs of conjugates, it also belongs to a whole pencil of quadrics of the web, given by the equation

$$\lambda_1 a_3 + \lambda_2 b_3 + \lambda_3 c_3 = 0$$

and the associated value of (X) is $(\lambda_2, -\lambda_1, 0, 0)$.

Hence the image of a point (y) *on any line in* (y) *is the whole line in* (X); *conversely, the image in* (y) *of any point* (X) *on one of the lines in* (X) *is the whole line in* (y).

5. **The fundamental lines** $l\,(X)$. We shall denote by l the lines joining P_8 to the other seven basis points of the net of quadrics. Consider the point (X) near $(0, 0, X_3, X_4)$. We have $\lambda_3 = 0$ and $\lambda_1 A(x) = \lambda_2 B(x) = 0$ for

the pencil of quadrics. The tangent plane to the quadric at (X) is $[(X_4+a_2X_3)+b_2X_3]x_1+[a_1X_3+(X_4+b_1X_3)]x_2=0$. If we eliminate $\lambda_1:\lambda_2$ we have a cubic cone on which lie the points (y) (y'); moreover

$$\frac{y_1}{y_2}=\frac{y_1'}{y_2'},$$

hence the image of the line $X_1=0$, $X_2=0$ is the quartic $y_1y_2'=y_1'y_2$, that is, $p_{12}=0$. The image in (y) of a point on this line $l(X)$ is a curve of order eight in which the cubic cone and the quadric meet apart from the line $y_1=0$, $y_2=0$. Similarly for each of the seven lines joining P_8 to each of the other seven basis points.

The image in (y) of any line joining P_8 to one of the other basis points P_i, considered as a line of (X) is a quartic surface, having the line defined by the same equations in (y) for double line, and containing the six lines through each of the basis points on the line which join it to the other basis points. Moreover, all of these seven quartic surfaces pass through the curve γ_6.

Any two of these quartic surfaces meet in γ_6, in their respective double lines, counting for four, in the other five lines $l(y)$, and in the line $m(y)$ joining the basis points on their double lines.

6. **The fundamental curve γ_6 in (X).** Any point (X) of γ_6 is the vertex of a quadric cone belonging to the net; every generator of this cone contains a pair of conjugate points (y) (y'), hence the image of (X) is a curve lying on its quadric cone. There are two positions of (y) the conjugates of which in I coincide with any given point on γ_6, because the conjugate of a point (X) on γ_6 is every point of a straight line, which is a bisecant of γ_6 and meets the cone in two points. Hence the image curve of (X) has a double point at (X). Since any plane passing through (X) meets the cone in two generators, each of which contains two image points, it follows that the curve is a sextic, having a double point at (X). The locus of this sextic curve, as (X) describes γ_6, is a surface, the order of which is determined as follows: the jacobian of the web of s_{12} is of order 44, and it consists of the seven quartic surfaces $p_{ik}=0$, and of the image of γ_6, which is therefore of order 16. This surface Γ_{16} has all the 28 lines joining basis points for double lines, contains γ_6 four-fold; the point P_8 is eight-fold, the other seven basis points being six-fold.

7. **Images of the plane $X_i=0$, $i=1,2,3$.** The image of the plane $X_1=0$ consists of the two quartics (82), (83), and of a residual quartic. Since (82) has the line P_8P_2 double, and the lines joining either P_8 or P_2 to the other six basis points simple, and similarly (83) contains the line P_8P_3 doubly, and the lines joining either P_8 or P_3 to each of the other basis points simply, and since the configuration of multiple lines is symmetric in all the lines on s_{12}, it follows that the residual quartic surface, image of the plane $X_1=0$, apart

from the fundamental lines in it, contains the lines joining P_8 to the other five basis points, the four lines joining the remaining vertex of the tetrahedron of reference to the four remaining basis points, and the six edges of the tetrahedron formed by the basis points not at the vertices of the tetrahedron of reference. Similarly for $X_2 = 0$, $X_3 = 0$.

We shall now prove analytically that the image of any plane of the bundle $k_1 X_1 + k_2 X_2 + k_3 X_3 = 0$ has the seven lines $l(y)$ triple and the twenty-one lines $m(y)$ simple. From the polar relations $A(y, y') = 0$, $B(y, y') = 0$, we have

$$y_4 y_1' + a_1 y_2' y_3 + a_2 y_3 y_1' + a_3 y_1 y_2' = -(y_4' y_2 + a_1 y_2 y_3' + a_2 y_3' y_1 + a_3 y_2 y_1'),$$

$$y_4' y_2 + b_1 y_2 y_3' + b_2 y_3' y_1 + b_3 y_1' y_2 = -(y_4 y_2' + b_1 y_2' y_3 + b_2 y_3 y_1' + b_3 y_2' y_1).$$

Multiplying the two equations together, member by member, then transposing the second member to the right, and subtracting the result from the value of $\lambda_3 = A(y)\,B(y') - A(y')\,B(y)$, we obtain

$$\lambda_3 = p_{12}[2 y_4 y_4' + (a_2 + b_1)(y_4 y_3' + y_3 y_4') + b_3(y_4 y_1' + y_4' y_1) + 2(a_1 b_2) y_3 y_3' + (a_2 b_3)(y_3 y_1' + y_3' y_1)].$$

Eliminating the terms $y_4 y_1' + y_4' y_1$ by means of $A(y, y') = 0$, we may write

$$\frac{\lambda_3}{p_{12}} = 2 y_4 y_4' + (a_2 + b_1)(y_4 y_3' + y_3 y_4') + 2(a_1 b_2) y_3 y_3' - a_3 b_2 (y_3 y_1' + y_3' y_1) - b_3 a_1 (y_3 y_2' + y_2 y_3') - a_3 b_3 (y_1 y_2' + y_2 y_1') = \mu_3.$$

Similar expressions can be found for $\mu_1 = \lambda_1/p_{23}$ and $\mu_2 = \lambda_2/p_{31}$, hence the correspondence (6) may now be written in the form

$$\begin{aligned} X_1 &= p_{12}\, p_{31}\, (\mu_2 - \mu_3), \\ X_2 &= p_{23}\, p_{12}\, (\mu_3 - \mu_1), \\ X_3 &= p_{31}\, p_{23}\, (\mu_1 - \mu_2), \\ X_4 &= p_{41}\, p_{23}\, \mu_1 + p_{42}\, p_{31}\, \mu_2 + p_{43}\, p_{12}\, \mu_3 \\ &= p_{41}\, p_{23}\, (\mu_1 - \mu_3) + p_{42}\, p_{31}\, (\mu_2 - \mu_3). \end{aligned} \tag{6'}$$

It is now readily verified that the seven central lines $l(y)$ are triple on the surfaces $X_1 = 0$, $X_2 = 0$, $X_3 = 0$, and double on $X_4 = 0$, while the twenty-one lines $m(y)$ are simple on all the surfaces of the web. The basis curve of order 63 is therefore made up of the seven central lines $l(y)$, each counted six times, and the 21 lines $m(y)$.

The tangent planes to the surfaces of the web at each point of the seven lines are fixed. Moreover, it can be shown that for the image of the plane $X_4 = 0$ the tangent cone at P_8 is of order six, while the tangent cone of the image of any plane of the bundle through $(0, 0, 0, 1)$ at P_8 is of order eight.

Hence the curve of order 27, image of a general line of (X), has six variable branches through P_8 lying on the tangent cone of the image of $X_4 = 0$. The image of the point $P_8(y)$ is therefore a surface of order 6 in (X).

Moreover, the image in (X) of a plane in (y) is a surface of order 27, having the seven central lines $l(X)$ eight-fold, since the image of a point of one of these lines is a curve of order 8 in (y). The curve γ_6 is 6-fold on all the surfaces of the system, for a similar reason.

This surface s'_{27} is the image of a plane in (y) and also of the conjugate cubic surface of the plane in the involution I; hence it follows that the image of the curve $\gamma_6(y)$ is a surface Γ'_{54} of order 54 in (X), having the seven central lines $l(X)$ 16-fold and the curve $\gamma_6(X)$ 12-fold. The complete image in (y) of this surface Γ'_{54} is the curve $\gamma_6(y)$ and its conjugate ruled surface of order 8 in I. Hence the image in (y) of a straight line in (X) is a curve of order 27, having a six-fold point at $P_8(y)$, a four-fold point at each of the other basis points and meeting $\gamma_6(y)$ in 54 points. The curve meets a general surface of the web s_{12} in $12 \cdot 27 - 4 \cdot 4 \cdot 7 - 6 \cdot 8 - 54 \cdot 3 = 2$ variable intersections. The point P_8 counts for 48 intersections on account of the fixed sextic cone at P_8.

This is verified by replacing the curve of order 27 by the image of a line through $P_8(X)$. Each of the seven central lines $l(y)$ now counts for 9 in the intersection of the image surfaces; hence the image of any such line is a curve of order 6 meeting γ_6 in 12 points and passing simply through the eight basis points P_i. The sextic curve meets the image of a general plane of (X) in $6 \cdot 12 - 32 \cdot 3 - 7 \cdot 4 - 6 = 2$ variable points. Since a line through $P_8(X)$ has a sextic curve for image, the image of P_8 itself consists of the seven central lines $l(y)$, each counted three times.

7. Using the notation previously employed, we may write

$$
\begin{aligned}
&s'_1 \backsim s_{12}\colon\ \gamma_6^3 + 7\,l + 21\,m,\\
&c'_1 \backsim c_{27}\colon\ P_8^6 + 7\,P_i^4,\ [c_{27}, \gamma_6] = 54,\\
&s_1 \backsim s'_{27}\colon\ \gamma_6^{16} + 7\,l'^8,\\
&c_1 \backsim c'_{12}\colon\ [c'_{12}, l'] = 4,\\
&\gamma_6 \backsim \Gamma'_{54}\colon\ \gamma_6'^{12} + 7\,l'^{16},\\
&\gamma'_6 \backsim \Gamma_{16}\colon\ \gamma_6^4 + P_8^8 + 7\,P_i^6 + 7\,l^2 + 21\,m^2,\\
&l \backsim A'_4\colon\ l^2 + 6\,l' + 6\,m' + \gamma'_6,\\
&P'_i \backsim (8\,i)_4\colon\ l_i^2 + 6\,l + 6\,m + \gamma_6,\\
&P_8 \backsim s'_6,\\
&X_i = 0 \backsim s_{12}\colon\ \gamma_6^3 + 7\,l^3 + 21\,m, \qquad i = 1, 2, 3,\\
&X_4 = 0 \backsim s_{12}\colon\ \gamma_6^3 + 7\,l^3 + 21\,m,\\
&P'_8 \backsim 7\,l^3.
\end{aligned}
$$

8. **The cubic inversion.** If the quadrics $A(x)=0$, $B(x)=0$, $C(x)=0$ have the special forms $x_1x_4+x_2x_3=0$, $x_2x_4+x_3x_4=0$, $x_3x_4+x_1x_2=0$, then the four basis points, apart from the vertices of the tetrahedron of reference, are $(1,1,1,-1)$, $(1,1,-1,1)$, $(1,-1,1,1)$, $(-1,1,1,1)$ respectively. This case can be discussed more easily if we change the coördinate system to the self-polar tetrahedron of the three quadrics. We may then take three quadrics of the form $x_1^2-x_4^2=0$, $x_2^2-x_4^2=0$, $x_3^2-x_4^2=0$, so that the net of quadrics has the form $\sum\lambda_i x_i^2=0$, where $\sum\lambda_i=0$.

The involution is now defined by the equations

$$y_1y_1'=y_2y_2'=y_3y_3'=y_4y_4'.$$

The equation of the quadric through (y) and (y') is given by $\sum\lambda_i y_i^2=0$, $\sum\lambda_i y_i'^2=0$ and is therefore

$$\begin{vmatrix} x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ y_1^2 & y_2^2 & y_3^2 & y_4^2 \\ y_1'^2 & y_2'^2 & y_3'^2 & y_4'^2 \\ 1 & 1 & 1 & 1 \end{vmatrix}=0.$$

The basis points are now $(1,\pm1,\pm1,\pm1)$. We shall take for P_8 the basis point $(1,1,1,1)$. The tangent plane at $(ky+k'y')$ passes through the point P_8 if $\sum\lambda_i(ky_i+k'y_i')=0$; hence

$$\frac{k}{k'}=-\frac{\sum\lambda_i y_i'}{\sum\lambda_i y_i},$$

so that

$$X_1=\lambda_2p_{12}+\lambda_3p_{13}+\lambda_4p_{14},$$

and similarly for X_2, X_3, X_4.

We find

$$\lambda_1=(y_2^2-y_3^2)(y_3^2-y_4^2)(y_4^2-y_2^2)y_1^2,$$

and similarly for $\lambda_2, \lambda_3, \lambda_4$, so that

$$k'=\prod(y_1-y_2)\sum y_1y_2y_3, \qquad k=-\prod(y_1-y_2)\sum y_1\cdot y_1y_2y_3y_4.$$

Hence

$$X_1=(y_1+y_2)(y_1+y_3)(y_1+y_4)y_2y_3y_4, \tag{7}$$

and similarly for X_2, X_3, X_4.

The correspondence is therefore of order 6, the images in (y) of the planes of (X) being surfaces of Enriques* having the six edges of the tetrahedron of reference for double curve. These edges constitute the γ_6 of the general case.

* See Picard et Simart, *Théorie des Fonctions algébriques de deux Variables indépendents*, vol. 2, p. 148.

The surfaces also contain the lines $y_1+y_2=0$, $y_3+y_4=0$, and two similar lines. Two surfaces of the web meet in a variable curve of order 9, image of a line in (X).

9. **Fundamental elements in** (y). The image in (X) of the point $(1,1,1,1)$ in (y) is clearly the point $(1,1,1,1)$. To obtain the image of another basis point, for example $(-1,1,1,1)$, consider a neighboring point $(-1, 1+\beta, 1+\gamma, 1+\delta)$. We at once find $X_1=0$. Similarly for the other three points of this type. Next, consider the point $(1, 1+\beta, -1+\gamma, -1+\delta)$ near the basis point $(1,1,-1,-1)$. Here we find $X_1=2\gamma\delta$, $X_2=2(\beta+\gamma)(\beta+\delta)$, $X_3=-2\gamma(\beta+\gamma)$, $X_4=-2\delta(\beta+\delta)$. Hence the image surface is $X_3X_4-X_1X_2=0$. Similarly for the other two points of this type.

The composite curve γ_6. To obtain the image in (X) of a line $y_1=0$, $y_2=0$, consider a point (y) near the line, and retain only the lowest powers of y_1 and y_2. Thus

$$X_1=y_2(y_1+y_2)y_3y_4, \qquad X_2=y_1(y_1+y_2)y_3y_4,$$
$$X_3=y_1y_2y_3(y_3+y_4), \qquad X_4=y_1y_2y_4(y_3+y_4),$$

hence by eliminating y_1 and y_2 we have the image of the point. The point $(0,0,y_3,y_4)$ has for image the curve $(y_3+y_4)X_1X_2-y_4X_3(X_1+X_2)=0$, $y_4X_3=y_3X_4$. The image of the whole line is obtained by eliminating y_3, y_4 from these equations. The image of the line $y_1=0$, $y_2=0$ and also of the line $y_3=0$, $y_4=0$ is therefore the cubic surface $X_1X_2(X_3+X_4)=X_3X_4(X_1+X_2)$. Similarly for the other four edges of the tetrahedron of reference.

The line $y_1+y_2=0$, $y_3+y_4=0$. Proceeding as before, take a point $(y_1, -y_1+p, y_3, -y_3+q)$ near the line and retain only the lowest powers of the small numbers p, q. After discarding a common factor we find

$$X_1=py_3, X_2=-py_3, X_3=-qy_1, X_4=qy_1,$$

so that $X_1+X_2=0$, $X_3+X_4=0$. The image of any point on the first line is therefore the whole of the second line. Similarly for the otner two basis lines in (y) having a similar form of equations.

The surfaces $s_6(y)$ of the web pass through the edges of the tetrahedron of reference doubly, contain the points $(1,1,-1,-1)$, etc. doubly, and the points, $(1,1,1,-1)$, etc. simply. Hence the complete image in (X) of any surface of the web consists of a plane $s_1'(X)$ counted twice, of six cubic surfaces, each taken twice, of three quadrics, each taken twice, and of the four planes $X_i=0$, each taken simply.

10. **Fundamental elements in** (X). *Vertices of the composite curve* γ_6. From the expressions for X_i in equations (7) in terms of (y) it is clear that

the point $(0, 0, 0, 1)$ in (y) has for image in (X) the point having the same coördinates. Conversely, therefore, the point $(0, 0, 0, 1)$ in (X) has one image at $(0, 0, 0, 1)$ in (y); the other is the entire plane $y_4 = 0$, since any plane of (X) through this point has for image a sextic surface having the fixed component $y_4 = 0$. The residual quintic surface meets $y_4 = 0$ in a composite quintic consisting of three lines of γ_6 and a variable conic. Hence the point $(0, 0, 0, 1)$ is 5-fold on s_9'.

11. **Lines of the composite curve γ_6.** The image of any plane of the pencil through the line $X_1 = 0$, $X_2 = 0$ has the plane $y_1 + y_2 = 0$ for fixed component in addition to the planes, images of the vertices on the line. Hence the plane is the image of the line. The sextic surface, image of a general plane in (X), meets the plane $y_1 + y_2 = 0$ in a cubic curve apart from fixed fundamental elements, hence the curve is the image of a point on the line $X_1 = 0$, $X_2 = 0$ and the line is triple on every s_9' of the system.

The point $(1, 1, 1, 1)$. The images of planes through the point $(1, 1, 1, 1)$ have in common in addition to fundamental elements common to all the sextic surfaces of the web, the three lines of the type $y_1 - y_2 = 0$, $y_3 - y_4 = 0$; hence the image of the point is these three lines.

The surfaces s_9' all have the vertices of the composite γ_6 for five-fold points, the edges three-fold. Moreover, they also have the lines of the type $X_1 - X_2 = 0$, $X_3 - X_4 = 0$ double. The image in (y) of an s_9' of the system is of order 54. It consists of s_1, its conjugate s_3 in I, and of the following fundamental elements: the four planes $y_i = 0$, each counted five times, the six planes $y_i + y_k = 0$, each counted three times, and three quadrics of the type $y_1 y_2 - y_3 y_4 = 0$, each counted twice.

We have seen that any point on the line $y_1 + y_2 = 0$, $y_3 + y_4 = 0$ has for image in (X) the whole line $X_1 + X_2 = 0$, $X_3 + X_4 = 0$. Since any plane in (y) meets the line in one point, its image s_9' contains the whole line. Hence the system s_9' has also the three lines of this type for basis elements.

12. **Algebraic procedure.** The plane containing the points (y), (y'), and the vertex $(1, 0, 0, 0)$ has the equation

$$p_{34} x_2 + p_{42} x_3 + p_{23} x_4 = 0.$$

Since (y) and (y') both satisfy this equation we may write

$$p_{34} y_2 + p_{42} y_3 + p_{23} y_4 = 0$$

and

$$\frac{p_{34}}{y_2} + \frac{p_{42}}{y_3} + \frac{p_{23}}{y_4} = 0;$$

hence

$$\frac{y_4}{y_3} + \frac{y_3}{y_4} + \frac{p_{23}^2 + p_{24}^2 - p_{34}^2}{p_{23}\, p_{34}} = 0.$$

Similarly, by considering the plane (y), (y'), (0, 1, 0, 0), we obtain

$$\frac{y_4}{y_3}+\frac{y_3}{y_4}+\frac{p_{13}^2+p_{14}^2-p_{34}^2}{p_{13}\,p_{14}}=0. \tag{8}$$

If we subtract the latter equation from the former, clear of fractions, and make use of the quadratic identity in (p), we obtain

$$p_{34}\,p_{42}\,p_{23}+p_{41}\,p_{13}\,p_{34}+p_{12}\,p_{24}\,p_{41}+p_{23}\,p_{31}\,p_{12}=0,$$

which is the equation of the cubic complex containing the lines (y), (y'). Conversely, given a line of the cubic complex, we can find a pair of points (y) from the above equations.

Given the point (X), the quadric $\sum\lambda_i x_i^2=0$, $\sum\lambda_i=0$ that contains the line (X) (1, 1, 1, 1) is given by $\sum\lambda_i X_i=0$, $\sum\lambda_i X_i^2=0$.

Hence

$$\lambda_1=(X_2-X_3)\,(X_3-X_4)\,(X_4-X_2), \text{ etc.}$$

By considering the intersection of the quadric and its tangent plane at (X) we find

$$p_{12}=(X_3-X_4)\,\{X_3\,X_4(X_1+X_2)-X_1\,X_2(X_3+X_4)\},$$

and similarly for the other line coördinates.

Given a plane $Ay_1+By_2+Cy_3+Dy_4=0$, we have $p_{34}y_2+p_{42}y_3+p_{23}y_4=0$ and three similar equations. Hence $y_1=Bp_{34}+Cp_{42}+Dp_{23}$ and three similar equations.

By substituting for (y') in this equation we have the image s_9' of the given plane in the form

$$-(A^2+B^2+C^2+D^2)\,p_{12}\,p_{13}\,p_{14}+\sum BC\,p_{14}\,(p_{12}^2+p_{13}^2-p_{14}^2)=0.$$

The factor $(X_3-X_4)(X_4-X_2)(X_2-X_3)$ divides out as can be seen by writing the equation in the form

$$-(A+B+C+D)^2\,p_{34}\,p_{42}\,p_{23}+\sum BC\,p_{23}\,[(p_{34}+p_{42})^2-p_{23}^2]=0,$$

but $(p_{34}+p_{42})-p_{23}=4\,X_1\,X_4\,(X_2\,X_3+X_1\,X_4)(X_2-X_3)(X_3-X_4)(X_4-X_2)$, hence the image of the plane is

$$-(A+B+C+D)^2\,QRS+BCX_1\,X_4(X_2\,X_3-X_1\,X_4)(X_2-X_3)^2S=0$$

in which Q, R, S are three cubic expressions of the form

$$X_3\,X_4(X_1+X_2)-X_1\,X_2(X_3+X_4), \text{ etc.}$$

The two values of y_3, y_4 are equal if

$$\frac{p_{24}^2 + p_{23}^2 - p_{34}^2}{p_{23}p_{24}} = \pm 2. \tag{9}$$

Hence

$$[(p_{23} + p_{24})^2 - p_{34}^2][(p_{23} - p_{24})^2 - p_{34}^2] = 0, \tag{10}$$

from which we find the surface of branch points to be

$$L' = X_1X_2X_3X_4(X_1X_4 - X_2X_3)(X_2X_4 - X_3X_1)(X_3X_4 - X_1X_2) = 0.$$

The component surfaces in $L' = 0$ are the images of the basis points $(1, \pm 1, \pm 1, \pm 1)$, apart from the point $(1, 1, 1, 1)$.

To express (y) in terms of (X) we start from equations similar to (8) but for y_2/y_1, y_3/y_1, and y_4/y_1. From these we find

$$\begin{aligned} y &= QRS, \\ y_2 + y_1 &= \{-2X_1X_2H_3H_4 + 2(X_1 - X_2)\sqrt{L'}\}Q, \\ y_3 + y_1 &= \{-2X_1X_3H_2H_4 + 2(X_1 - X_3)\sqrt{L'}\}R, \\ y_4 + y_1 &= \{-2X_1X_4H_2H_3 + 2(X_1 - X_4)\sqrt{L'}\}S, \end{aligned} \tag{11}$$

where H_z stands for $X_1X_2 - X_3X_4$, etc.

The equations (8) and (11) express algebraically the $(1, 2)$ correspondence between the spaces (X) and (y).

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SUR CERTAINES ÉQUATIONS AUX DIFFÉRENCES FINIES*

PAR

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I.

1. Pour abréger nous désignerons la différence du premier ordre avec l'intervalle ω par le symbole $\underset{\omega}{\triangle}$

$$\underset{\omega}{\triangle} F(x) = \frac{F(x+\omega)-F(x)}{\omega},$$

et la moyenne entre les valeurs d'une fonction dans les points x et $x+\omega$ par le symbole $\underset{\omega}{\triangledown}$

$$\underset{\omega}{\triangledown} F(x) = \frac{F(x+\omega)+F(x)}{2}.$$

Soient $\omega_1, \omega_2, \ldots, \omega_n$ des nombres positifs quelconques. Je définis la différence d'ordre n par l'équation

$$\underset{\omega_1 \ldots \omega_n}{\overset{n}{\triangle}} F(x) = \underset{\omega_n}{\triangle} \left(\underset{\omega_1 \ldots \omega_{n-1}}{\overset{n-1}{\triangle}} F(x) \right)$$

et la moyenne d'ordre n par l'équation

$$\underset{\omega_1 \ldots \omega_n}{\overset{n}{\triangledown}} F(x) = \underset{\omega_n}{\triangledown} \left(\underset{\omega_1 \ldots \omega_{n-1}}{\overset{n-1}{\triangledown}} F(x) \right).$$

En particulier, si $\omega_1 = \omega_2 = \cdots = \omega_n = \omega$, j'écris plus brièvement $\underset{\omega}{\overset{n}{\triangle}} F(x)$ et $\underset{\omega}{\overset{n}{\triangledown}} F(x)$. On a donc

$$\omega^n \underset{\omega}{\overset{n}{\triangle}} F(x) = \sum_{s=0}^{s=n} (-1)^{n-s} \binom{n}{s} F(x+s\omega),$$

$$2^n \underset{\omega}{\overset{n}{\triangledown}} F(x) = \sum_{s=0}^{s=n} \binom{n}{s} F(x+s\omega).$$

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* Presented to the Society, April 28, 1923.

Dans ce mémoire je vais étudier les solutions des deux équations suivantes

$$\overset{n}{\underset{\omega_1 \dots \omega_n}{\triangle}} F(x) = \varphi(x), \tag{1}$$

$$\overset{n}{\underset{\omega_1 \dots \omega_n}{\nabla}} G(x) = \varphi(x), \tag{2}$$

$\varphi(x)$ étant une fonction donnée qui, pour $x \geqq a$, admet une dérivée continue d'un certain ordre, soit d'ordre m, telle que

$$\lim_{x \to \infty} x^{n+\varepsilon} \varphi^{(m)}(x) = 0,$$

ε étant un nombre positif aussi petit que l'on veut. Ces deux équations admettent évidemment une infinité de solutions. Parmi elles j'en distingue une que j'appelle la *solution principale* et je me borne à étudier cette solution particulière. On obtient la solution la plus générale en ajoutant à la solution principale une fonction quelconque $\pi(x)$ qui satisfait à l'équation

$$\overset{n}{\underset{\omega_1 \dots \omega_n}{\triangle}} \pi(x) = 0,$$

respectivement à l'équation

$$\overset{n}{\underset{\omega_1 \dots \omega_n}{\nabla}} \pi(x) = 0.$$

Posons pour abréger

$$\Omega = s_1 \omega_1 + s_2 \omega_2 + \cdots + s_n \omega_n$$

et considérons la série à n entrées

$$2^n \sum (-1)^{s_1 + s_2 \dots + s_n} \varphi(x + \Omega) e^{-\eta(x+\Omega)}, \tag{3}$$

la sommation étant étendue à toutes les valeurs entières, non négatives de $s_1, s_2, \dots, s_n$. Cette série converge* évidemment pour toute valeur positive de η et nous allons démontrer que la somme de la série tend uniformément vers une limite quand le nombre positif η tend vers zéro. Cette limite représente, pour $x \geqq a$, une fonction continue de x, soit $G_n(x | \omega_1, \omega_2, \dots, \omega_n)$,

* En effet, la condition que nous avons imposée à la fonction $\varphi(x)$ entraîne qu'on sait trouver un nombre positif p tel que $\lim_{x \to \infty} \frac{\varphi(x)}{x^p} = 0$.

et l'on vérifie sans peine que cette fonction satisfait à l'équation (2). On aura donc

$$(4)\quad G_n(x\,|\,\omega_1, \omega_2, \ldots, \omega_n) = \lim_{\eta \to 0} 2^n \sum (-1)^{s_1+s_2\cdots+s_n} \varphi(x+\mathfrak{Q}) e^{-\eta(x+\mathfrak{Q})}$$

et cette solution sera par définition la solution principale de l'équation (2). D'une manière semblable je définis la solution principale de l'équation (1) que je désignerais par $F_n(x\,|\,\omega_1, \omega_2, \ldots, \omega_n)$. En donnant à n successivement les valeurs $1, 2, 3, \ldots$, on obtient ainsi deux suites de transcendantes nouvelles: $F_1, F_2, F_3, \ldots$ d'une part et $G_1, G_2, G_3, \ldots$ d'autre part. Le but de ce mémoire est de mettre en évidence les propriétés les plus importantes de ces transcendantes en les considérant comme des fonctions de x et de $\omega_1, \omega_2, \ldots, \omega_n$. J'ai déjà consacré un mémoire* étendu à l'étude du cas $n = 1$ et j'ai fait récemment un historique† complet de la théorie des équations aux différences finies. En renvoyant le lecteur à cet historique je me borne à mentionner ici qu'un cas particulier de l'équation (1) a été étudié par Barnes.‡

Sur une formule sommatoire

2. Dans ce qui suit nous ferons souvent usage de certaines séries multiples. Précisons d'abord ce que nous voulons entendre par une série convergente. Considérons la série à n entrées

$$(5)\quad \sum \varphi(s_1, s_2, \ldots, s_n),$$

la sommation étant étendue à toutes les valeurs entières non négatives de $s_1, s_2, \ldots, s_n$. Cette série est convergente si la somme

$$(6)\quad \sum_{s_n=0}^{s_n=p_n} \cdots \sum_{s_2=0}^{s_2=p_2} \sum_{s_1=0}^{s_1=p_1} \varphi(s_1, s_2, \ldots, s_n)$$

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* *Mémoire sur le calcul aux différences finies,* Acta Mathematica, vol. 44 (1923), pp. 71–211.

† *Sur l'état actuel de la théorie des équations aux différences finies,* Bulletin des Sciences Mathématiques, sér. 2, vol. 44 (1920), pp. 172–192, pp. 200–220; *Neuere Untersuchungen über Differenzengleichungen, Encyklopädie der mathematischen Wissenschaften,* II C 7 (1923).

‡ E. W. Barnes, *Genesis of the double Gamma Functions,* Proceedings of the London Mathematical Society, vol. 31 (1900), pp. 358–381; *The theory of the double Gamma Function,* Royal Society Philosophical Transactions, vol. 196 A (1901), pp. 265–387; *On the theory of the multiple Gamma Function,* Transactions of the Cambridge Philosophical Society, vol. 19 (1904), pp. 374–425. Voir aussi G. H. Hardy, *On the expression of the double Zeta-function and double Gamma-function in terms of elliptic functions,* Transactions of the Cambridge Philosophical Society, vol. 20 (1905), pp. 1–35.

tend vers une limite quand les entiers $p_1, p_2, \ldots, p_n$ tendent simultanément, mais indépendamment l'un de l'autre, vers l'infini, et cette limite s'appelle la somme de la série. De même considérons l'intégrale

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \varphi(t_1, t_2, \ldots, t_n)\, dt_1\, dt_2 \ldots dt_n,$$

φ étant continue pour les valeurs positives des variables. Cette intégrale est convergente si l'intégrale

$$\int_0^{p_n} dt_n \cdots \int_0^{p_2} dt_2 \int_0^{p_1} \varphi(t_1, t_2, \ldots, t_n)\, dt_1$$

tend vers une limite quand $p_1, p_2, \ldots, p_n$ tendent vers l'infini simultanément, mais indépendamment l'un de l'autre. Plusieurs auteurs* ont adopté une autre définition qui est telle que toute série (intégrale) multiple, qui converge, est absolument convergente. Il n'y a pas lieu de s'imposer cette restriction.† Les séries (intégrales) qui convergent, sans être absolument convergentes, peuvent rendre des services considérables en analyse et il n'y a aucune raison pour ne pas en tirer parti.

Si dans l'expression (6) on fait tendre d'abord p_1, puis $p_2, \ldots$ et enfin p_n vers l'infini, on obtient la série n fois itérée

$$\sum_{s_n=0}^{\infty} \cdots \sum_{s_2=0}^{\infty} \sum_{s_1=0}^{\infty} \varphi(s_1, s_2, \ldots, s_n).$$

Il convient de remarquer que cette série est essentiellement distincte de la série (5), la convergence de l'une n'entraîne pas nécessairement la convergence de l'autre.

Dans le but de démontrer que l'expression (3) tend vers une limite quand $\eta \to 0$, je commence par déduire une certaine formule sommatoire qui pourra d'ailleurs aussi rendre des services quand il s'agira d'étudier si une série multiple converge ou non.

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* Voir par exemple Jordan, *Cours d'Analyse*, 2e édition, vol. 1, p. 302; vol. 2, p. 78.

† Cf. les remarques de Pringsheim, Münchener Sitzungsberichte, vol. 27 (1897), pp. 139–141, et de Hardy, Messenger of Mathematics, ser. 2, vol. 32 (1903), pp. 92–97.

Soit $E_\nu(x)$ le polynome d'Euler* c'est-à-dire le polynome qui satisfait à l'équation

$$E_\nu(x+1)+E_\nu(x)=2x^\nu.$$

Soit $\bar{E}_\nu(x)$ une fonction qui satisfait à l'équation

$$\bar{E}_\nu(x+1)+\bar{E}_\nu(x)=0$$

et qui est égale au polynome $E_\nu(x)$ dans l'intervalle $0\leqq x<1$. Soit $\varphi(z)$ une fonction qui admet une dérivée d'ordre m qui est absolument continue dans l'intervalle $x\leqq z\leqq x+1$.

On aura†

$$(7)\quad \varphi(x+h)=\sum_{\nu=0}^{\nu=m}\frac{E_\nu(h)}{\nu!}\nabla\varphi^{(\nu)}(x)-\tfrac{1}{2}\int_0^1\frac{\bar{E}_m(h+t)}{m!}\varphi^{(m+1)}(x+1-t)\,dt,$$

h étant un nombre tel que $0\leqq h\leqq 1$. C'est ce qu'on voit aisément en intégrant par partie dans le dernier terme au second membre.

Plus généralement soit $E_\nu^{(n)}(x)$ le polynome d'Euler d'ordre n, c'est-à-dire le polynome qui satisfait à l'équation

$$(8)\quad \overset{n}{\underset{\omega_1\ldots\omega_n}{\nabla}}E_\nu^{(n)}(x)=x^\nu.$$

Soit $\varphi(x)$ un polynome quelconque du degré m. On a alors la relation identique‡

$$\varphi(x+h)=\sum_{\nu=0}^{\nu=m}\frac{E_\nu^{(n)}(h)}{\nu!}\overset{n}{\underset{\omega_1\ldots\omega_n}{\nabla}}\varphi^{(\nu)}(x),$$

h étant un nombre quelconque. Quelle pourra être la différence entre le premier et le second membre de cette équation si la fonction $\varphi(x)$ cesse d'être

* Au sujet des polynomes d'Euler et des polynomes de Bernoulli voir: *Mémoire sur les polynomes de Bernoulli*, Acta Mathematica, vol. 43 (1920), pp. 121–196. Dans les pages suivantes nous nous servirons souvent des résultats obtenus dans ce mémoire.

† *Mémoire sur le calcul aux différences finies*, Acta Mathematica, vol. 44 (1923), p. 88. L'équation (7) a été indiquée, sans terme reste, par Euler dans le cas particulier $h=0$. Elle a été retrouvée par Oettinger (*Die Lehre von den aufsteigenden Functionen*, 1836, p. 44) et plus tard par Boole et on la désigne quelquefois comme la formule sommatoire de Boole.

‡ Acta Mathematica, vol. 43, p. 156.

un polynome? Pour le voir je définis, de la manière suivante, une fonction que je désigne par $\bar{E}_\nu^{(n)}(x|\,\omega_1, \omega_2, \ldots, \omega_n)$ ou plus brièvement par $\bar{E}_\nu^{(n)}(x)$. Cette fonction doit satisfaire à l'équation

$$\overset{n}{\underset{\omega_1 \ldots \omega_n}{\nabla}} \bar{E}_\nu^{(n)}(x) = 0 \tag{9}$$

et l'on a en outre dans l'intervalle $0 \leqq x < \omega_1 + \omega_2 + \cdots + \omega_n$

$$\bar{E}_\nu^{(n)}(x|\,\omega_1, \ldots, \omega_n) = E_\nu^{(n)}(x|\,\omega_1, \ldots, \omega_n).$$

La fonction est uniquement définie, pour toutes les valeurs réelles de x, par les deux conditions que nous venons de lui imposer. Elle est continue à l'intérieur de l'intervalle $0 < x < \omega_1 + \omega_2 + \cdots + \omega_n$. Si $\nu > 0$ elle est encore continue dans le point $x = \omega_1 + \omega_2 + \cdots + \omega_n$. En effet, la fonction est définie dans ce point par l'équation (9), où l'on pose $x = 0$. Mais comme le second membre de l'équation (8) tend vers zéro avec x notre fonction prend la même valeur que le polynome d'Euler dans le point en question. Elle est par conséquent continue dans ce point. Cela posé considérons l'équation (9) et faisons varier x en partant du point $x = 0$. On obtient successivement, pour toutes les valeurs réelles de x, la valeur de la fonction $\bar{E}_\nu^{(n)}(x)$ comme une somme de $2^n - 1$ fonctions continues. Notre fonction est par conséquent continue partout, si $\nu > 0$. Mais cela n'est plus vrai si $\nu = 0$ parce que le second membre de l'équation (8) se réduit à *un*. Il en résulte que la fonction $\bar{E}_0^{(n)}(x)$ est discontinue dans le point $x = 0$ et par conséquent dans une infinité d'autres points que nous énumérons plus loin.

Nous allons indiquer quelques propriétés importantes de la fonction $\bar{E}_\nu^{(n)}(x)$. On aura

$$\bar{E}_\nu^{(n)}(\omega_1 + \omega_2 + \cdots + \omega_n - x) = (-1)^\nu \bar{E}_\nu^{(n)}(x), \qquad (\nu > 0). \tag{10}$$

Car des propriétés du polynome d'Euler* il résulte que cette relation est vraie dans l'intervalle $0 \leqq x \leqq \omega_1 + \omega_2 + \cdots + \omega_n$. En outre la moyenne $n^{\text{ième}}$ de la fonction au premier membre est égale à zéro. C'est ce qu'on voit en remplaçant x par $-x$ dans l'équation (9). La relation (10) subsiste donc pour toutes les valeurs réelles de x. On voit de même que

$$\frac{d\bar{E}_\nu^{(n)}(x)}{dx} = \nu \bar{E}_{\nu-1}^{(n)}(x), \qquad (\nu > 0). \tag{11}$$

* Acta Mathematica, vol. 43, p. 157.

Car cette relation est vraie dans l'intervalle $0 < x < \omega_1 + \omega_2 + \cdots + \omega_n$ et les deux membres ont leur moyenne $n^{\text{ième}}$ égale à zéro. Elle est donc vraie partout. Si $\nu = 0$ la dérivée cesse d'exister dans une infinité de points. $\overline{E}_\nu^{(n)}(x)$ est donc, si $\nu > 0$, une fonction continue de x qui admet des dérivées continues des ordres $1, 2, \ldots, \nu - 1$. Elle admet en outre une dérivée d'ordre ν qui est discontinue dans une infinité de points.

La fonction $\overline{E}(x)$ satisfait à l'équation

$$\bigtriangledown_{\omega_n} \overline{E}_\nu^{(n)}(x) = \overline{E}_\nu^{(n-1)}(x \mid \omega_1, \omega_2, \ldots, \omega_{n-1}). \tag{12}$$

Car des propriétés du polynome d'Euler il résulte que cette équation est vraie dans l'intervalle $0 \leqq x < \omega_1 + \omega_2 + \cdots + \omega_{n-1}$. Et, par définition, les moyennes $(n-1)^{\text{ièmes}}$ des fonctions au premier et au second membres de l'équation (12) sont nul. Ces deux fonctions sont donc égales pour toutes les valeurs de x.

Considérons en particulier le cas $\nu = 0$. La fonction $\overline{E}_0^{(1)}(x \mid \omega_1)$ est définie par les conditions

$$\overline{E}_0^{(1)}(x + \omega_1) = -\overline{E}_0^{(1)}(x),$$
$$\overline{E}_0^{(1)}(x) = 1, \qquad (0 \leqq x < \omega_1).$$

Elle est égale à $+1$ ou à -1. Elle est donc continue sauf dans les points $x = 0, \pm\omega_1, \pm 2\omega_1, \ldots$ où elle passe de ± 1 à ∓ 1.

Considérons la fonction $\overline{E}_0^{(2)}(x \mid \omega_1, \omega_2)$. Par définition elle est continue dans l'intervalle $0 < x < \omega_1 + \omega_2$ et l'on aura

$$\overline{E}_0^{(2)}(x + \omega_2) + \overline{E}_0^{(2)}(x) = 2\,\overline{E}_0^{(1)}(x \mid \omega_1).$$

Il en résulte que notre fonction est discontinue dans les points $p_1\omega_1 + p_2\omega_2$, p_1 et p_2 étant des entiers positifs, et elle y fait un saut brusque égal à $4(-1)^{p_1+p_2+1}$. En général, elle est continue dans les points $p_1\omega_1$ et dans les points $p_2\omega_2$. De plus notre fonction est discontinue dans les points $-p_1\omega_1 - p_2\omega_2$, p_1 et p_2 étant des entiers non négatifs, et elle y fait un saut brusque égal à $4(-1)^{p_1+p_2}$. Dans tout autre point la fonction est continue.

Par voie d'induction on démontre que la fonction $\overline{E}_0^{(n)}(x \mid \omega_1, \ldots, \omega_n)$, qui est égale à un dans l'intervalle $0 \leqq x < \omega_1 + \omega_2 + \cdots + \omega_n$, est discontinue dans les points $p_1\omega_1 + p_2\omega_2 + \cdots + p_n\omega_n$ de telle manière que

$$\overline{E}_0^{(n)}(p_1\omega_1 + p_2\omega_2 + \cdots + p_n\omega_n + 0) - \overline{E}_0^{(n)}(p_1\omega_1 + p_2\omega_2 + \cdots + p_n\omega_n - 0)$$
$$= 2^n(-1)^{p_1+p_2+\cdots+p_n+n-1},$$

$p_1, p_2, \ldots, p_n$ étant des entiers positifs. Et elle est discontinue dans les points $-p_1\omega_1 - p_2\omega_2 - \cdots - p_n\omega_n$ de telle manière que

$$\bar{E}_0^{(n)}(-p_1\omega_1 - p_2\omega_2 - \cdots - p_n\omega_n + 0) - \bar{E}_0^{(n)}(-p_1\omega_1 - p_2\omega_2 - \cdots - p_n\omega_n - 0) = 2^n(-1)^{p_1+p_2+\cdots+p_n},$$

$p_1, p_2, \ldots, p_n$ étant des entiers non négatifs. Dans tout autre point la fonction est continue et elle est constante dans tout intervalle limité par deux points de discontinuité consécutifs.

S'il arrive que q des points susdits viennent coïncider dans un point nous comptons ce point comme un point de discontinuité q^{iple} et dans un tel point le saut est q fois plus grand que dans un point simple.

Ce fait se présente en particulier si tous les nombres $\omega_1, \omega_2, \ldots, \omega_n$ se réduisent à un. En ce cas on sait trouver une expression bien simple de notre fonction. Je dis qu'on a

$$(13)\quad \bar{E}_\nu^{(n+1)}(x) = \frac{2^n}{n!}\sum_{s=0}^{s=n}\frac{(-1)^{n-s}}{s!}\bar{E}_{\nu+s}(x)\,D_x^s(x-1)(x-2)\cdots(x-n).$$

En effet, en appliquant l'opération ∇ aux deux membres de l'équation (13) il vient

$$\nabla\,\bar{E}_\nu^{(n+1)}(x) = \bar{E}_\nu^{(n)}(x).$$

D'autre part, des propriétés des polynomes d'Euler* il résulte que l'équation (13) est vraie dans l'intervalle $0 \leqq x < 1$. Par voie d'induction on démontre alors qu'elle est vraie pour toutes les valeurs de x.

De cette équation il résulte en particulier que

$$\bar{E}_0^{(n+1)}(x) = \frac{(-2)^n}{n!}(x-1)(x-2)\cdots(x-n)\,\bar{E}_0(x) + f(x),$$

$f(x)$ étant une fonction continue de x. On voit donc que la fonction $\bar{E}_0^{(n)}(x)$ est discontinue dans les points $x = n, n+1, n+2, \ldots$ et dans les points $x = 0, -1, -2, -3, \ldots$ et il n'y a pas d'autres points de discontinuité. Si p est un entier positif, nul ou négatif on aura

$$\bar{E}_0^{(n)}(p+0) - \bar{E}_0^{(n)}(p-0) = (-1)^{p+n-1}2^n\frac{(p-1)(p-2)\cdots(p-n+1)}{1\cdot 2\cdot 3\cdots(n-1)}.$$

* Acta Mathematica, vol. 43, p. 188.

Ce résultat est en accord avec celui que nous venons de trouver dans le cas général.

Comment se comporte la fonction $\overline{E}_\nu^{(n)}(x\,|\,\omega_1, \omega_2, \ldots, \omega_n)$ quand $|x|$ tend vers l'infini? Si les ω_i satisfont à certaines conditions cette fonction reste comprise entre des limites finies. Je me borne à indiquer une limite supérieure du module de la fonction, valable quels que soient les ω_i. On a d'abord

$$(-1)^{p_2}\,\overline{E}_\nu^{(2)}(x+p_2\,\omega_2) = \overline{E}_\nu^{(2)}(x) - 2\sum_{s=0}^{p_2-1}(-1)^s\,\overline{E}_\nu^{(1)}(x+s\,\omega_2\,|\,\omega_1).$$

Comme $\overline{E}_\nu^{(1)}(x)$ est périodique et bornée on en conclut que

$$\frac{\overline{E}_\nu^{(2)}(x\,|\,\omega_1, \omega_2)}{x}$$

reste comprise entre des limites finies quand $|x|$ tend vers l'infini. En tenant compte de l'équation (12) on démontre par voie d'induction que

$$\frac{\overline{E}_\nu^{(n)}(x\,|\,\omega_1, \ldots, \omega_n)}{x^{n-1}}$$

reste comprise entre des limites finies quand $|x|$ tend vers l'infini. Il arrive que cette expression tend vers zéro mais ce n'est pas toujours le cas. Par exemple si tous les ω_i se réduisent à *un* l'équation (13) montre que $\overline{E}_\nu^{(n)}(x)$ est de la forme

$$\frac{\overline{E}_\nu^{(n)}(x)}{x^{n-1}} = \pi(x) + \varepsilon(x),$$

$\pi(x)$ étant une fonction périodique et bornée, qui n'est pas identiquement zéro, $\varepsilon(x)$ étant une fonction qui tend vers zéro quand x tend vers l'infini.

3. Cela posé, soit $\varphi(z)$ une fonction qui admet une dérivée continue d'ordre $m+1$ dans l'intervalle $x \leqq z \leqq x+\omega_1+\omega_2+\cdots+\omega_n$, et considérons l'intégrale

$$(14) \qquad Q_m^{(n)} = -\tfrac{1}{2}\int_0^{\omega_n} \frac{\overline{E}_m^{(n)}(h+t)}{m!} \overset{n-1}{\underset{\omega_1 \cdots \omega_{n-1}}{\nabla}} \varphi^{(m+1)}(x+\omega_n - t)\,dt,$$

h étant un nombre situé dans l'intervalle $0 \leqq h < \omega_1 + \omega_2 + \cdots + \omega_n$. En intégrant par partie on trouve, si $m > 0$,

$$Q_m^{(n)} = Q_{m-1}^{(n)} + \frac{\bar{E}_m^{(n)}(h+\omega_n)}{2 \cdot m!} \underset{\omega_1 \cdots \omega_{n-1}}{\overset{n-1}{\nabla}} \varphi^{(m)}(x) - \frac{\bar{E}_m^{(n)}(h)}{2 \cdot m!} \underset{\omega_1 \cdots \omega_{n-1}}{\overset{n-1}{\nabla}} \varphi^{(m)}(x+\omega_n),$$

mais cette relation peut s'écrire

$$(15) \quad Q_m^{(n)} = Q_{m-1}^{(n)} - \frac{\bar{E}_m^{(n)}(h)}{m!} \underset{\omega_1 \cdots \omega_n}{\overset{n}{\nabla}} \varphi^{(m)}(x) + \frac{\bar{E}_m^{(n-1)}(h)}{m!} \underset{\omega_1 \cdots \omega_{n-1}}{\overset{n-1}{\nabla}} \varphi^{(m)}(x).$$

Si $n = 1$ le dernier terme au second membre se reduit à zéro parce que $\bar{E}_m^{(1)}(h+\omega_1) + \bar{E}_m^{(1)}(h) = 0$. Posons

$$(16) \qquad R_m^{(n)}(x) = Q_m^{(n)} + Q_m^{(n-1)} + \cdots + Q_m^{(1)}.$$

On aura en vertu de l'équation (15)

$$R_m^{(n)}(x) = R_{m-1}^{(n)}(x) - \frac{\bar{E}_m^{(n)}(h)}{m!} \underset{\omega_1 \cdots \omega_n}{\overset{n}{\nabla}} \varphi^{(m)}(x).$$

Par conséquent

$$(17) \qquad R_m^{(n)}(x) = R_0^{(n)}(x) - \sum_{\nu=1}^{\nu=m} \frac{\bar{E}_\nu^{(n)}(h)}{\nu!} \underset{\omega_1 \cdots \omega_n}{\overset{n}{\nabla}} \varphi^{(\nu)}(x).$$

Il nous reste à évaluer $R_0^{(n)}(x)$. Dans ce but considérons l'intégrale $Q_0^{(n)}$ et supposons pour un moment que $0 \leqq h < \omega_1$. Si $n > 1$, la fonction $\bar{E}_0^{(n)}(h+t)$ est constante et égale à 1 dans l'intervalle d'intégration. On trouve par conséquent

$$(18) \qquad Q_0^{(n)} = \underset{\omega_1 \cdots \omega_{n-1}}{\overset{n-1}{\nabla}} \varphi(x) - \underset{\omega_1 \cdots \omega_n}{\overset{n}{\nabla}} \varphi(x).$$

Mais cette équation n'est plus vraie si $n = 1$. La fonction $\bar{E}_0^{(1)}(h+t)$ est en effet discontinue dans le point $t = \omega_1 - h$ qui est situé dans l'intervalle d'intégration. En tenant compte de ce fait on trouve

$$(19) \qquad Q_0^{(1)} = - \underset{\omega_1}{\nabla} \varphi(x) + \varphi(x+h).$$

De ces équations on déduit la valeur de $R_0^{(n)}(x)$:

$$(20)\quad R_0^{(n)}(x) = Q_0^{(n)} + Q_0^{(n-1)} + \cdots + Q_0^{(1)} = -\underset{\omega_1 \cdots \omega_n}{\overset{n}{\nabla}} \varphi(x) + \varphi(x+h).$$

Nous venons de supposer que h est situé dans l'intervalle $0 \leqq h < \omega_1$. Si h croît et sort de cet intervalle, les relations (18) et (19) cessent d'être vraies. Les fonctions $\overline{E}_0^{(2)}(h+t), \ldots, \overline{E}_0^{(n)}(h+t)$ deviennent en effet discontinues dans un nombre fini de points situés dans les intervalles d'intégration. Mais en se reportant à ce que nous avons dit dans le paragraphe précédent on vérifie sans peine que les termes qui proviennent de ces discontinuités se détruisent mutuellement de sorte que l'équation (20) reste vraie pourvu que $0 \leqq h < \omega_1 + \omega_2 + \cdots + \omega_n$.

Substituons maintenant l'expression (20) dans l'équation (17). On trouve la formule que j'avais en vue

$$(21)\quad \varphi(x+h) = \sum_{\nu=0}^{\nu=m} \frac{\overline{E}_\nu^{(n)}(h)}{\nu!} \underset{\omega_1 \cdots \omega_n}{\overset{n}{\nabla}} \varphi^{(\nu)}(x) + R_m^{(n)}(x),$$

où $0 \leqq h < \omega_1 + \omega_2 + \cdots + \omega_n$. Soit en particulier $h = 0$, il vient*

$$(22)\quad \varphi(x) = \sum_{\nu=0}^{\nu=m} \frac{C_\nu^{(n)}}{\nu!\, 2^\nu} \underset{\omega_1 \cdots \omega_n}{\overset{n}{\nabla}} \varphi^{(\nu)}(x) + R_m^{(n)}(x),$$

où

$$(23)\quad R_m^{(n)}(x) = -\tfrac{1}{2} \sum_{p=1}^{p=n} \int_0^{\omega_p} \frac{\overline{E}_m^{(p)}(t)}{m!} \underset{\omega_1 \cdots \omega_{p-1}}{\overset{p-1}{\nabla}} \varphi^{(m+1)}(x + \omega_p - t)\, dt.$$

Mais si l'on pose $h = \frac{1}{2}(\omega_1 + \omega_2 + \cdots + \omega_n)$ on trouve

$$(24)\quad \varphi\left(x + \frac{\omega_1 + \omega_2 + \cdots + \omega_n}{2}\right) = \sum_{\nu=0}^{\nu=m} \frac{E_{2\nu}^{(n)}}{(2\nu)!\, 2^{2\nu}} \underset{\omega_1 \cdots \omega_n}{\overset{n}{\nabla}} \varphi^{(2\nu)}(x) + R_{2m}^{(n)}(x),$$

---

* Nous avons supposé que les ω_i sont des nombres positifs parce que la fonction $\overline{E}(x \mid \omega_1, \ldots, \omega_n)$ n'est définie qu'en ce cas. Mais dans l'équation (22) la fonction $\overline{E}(x)$ a été remplacée par le polynome d'Euler et l'on n'a plus besoin de faire cette hypothèse. La démonstration que nous venons de donner de l'équation (22) reste valable quand on donne aux variables $x, \omega_1, \omega_2, \ldots, \omega_n$ des valeurs complexes quelconques pourvu que la dérivée $\varphi^{(m+1)}(x)$ reste continue pour les valeurs qui entrent en considération.

où

$$(25)\qquad R_{2m}^{(n)}(x) = -\tfrac{1}{2}\sum_{p=1}^{p=n}\int_0^{\omega_p}\frac{\bar{E}_{2m}^{(p)}\left(t+\frac{\omega_1+\cdots+\omega_n}{2}\right)}{(2m)!}\underset{\omega_1\cdots\omega_{p-1}}{\overset{p-1}{\nabla}}\varphi^{(2m+1)}(x+\omega_p-t)\,dt.$$

4. Le premier terme au second membre de l'équation (21) est une fonction symétrique des paramètres $\omega_1, \omega_2, \ldots, \omega_n$; il faut donc qu'il en soit de même du reste, mais l'expression de $R_m^{(n)}(x)$ que nous venons de trouver est apparemment asymétrique et pour cette raison peu commode dans certaines applications. On peut écrire le reste sous une autre forme qui est quelquefois préférable mais on doit alors faire une nouvelle hypothèse relativement à la fonction φ. Supposons que la dérivée $\varphi^{(m+1)}(z)$ est continue, pour $z \geqq x$, et que l'intégrale

$$(26)\qquad \int_0^{\infty}\bar{E}_m^{(n)}(-t)\,\varphi^{(m+1)}(z+t)\,dt$$

converge.* Dans ces conditions je dis qu'on a

$$(27)\qquad R_m^{(n)}(x) = \int_0^{\infty}\frac{\bar{E}_m^{(n)}(h-t)}{m!}\underset{\omega_1\cdots\omega_n}{\overset{n}{\nabla}}\varphi^{(m+1)}(x+t)\,dt.$$

De la convergence de l'intégrale (26) il résulte d'abord que toutes les intégrales

$$\int_0^{\infty}\bar{E}_m^{(p)}(-t)\,\varphi^{(m+1)}(z+t)\,dt \qquad (p=1,2,\ldots,n)$$

convergent pour $z \geqq x$. Si $n=1$ l'équation (16) se réduit à

$$R_m^{(1)}(x) = -\tfrac{1}{2}\int_0^{\omega_1}\frac{\bar{E}_m^{(1)}(h+\omega_1-t)}{m!}\varphi^{(m+1)}(x+t)\,dt.$$

Mais puisque

$$\underset{\omega_1}{\nabla}\bar{E}_m^{(1)}(x) = 0,$$

* Cette intégrale est en particulier absolument convergente si $\lim_{x\to\infty} x^{m+\varepsilon}\varphi^{(m+1)}(x) = 0$, ε étant un nombre positif.

cette équation peut s'écrire comme il suit:

$$R_m^{(1)}(x) = -\tfrac{1}{2}\int_0^{\omega_1} \frac{\bar{E}_m^{(1)}(h+\omega_1-t)}{m!}\,\varphi^{(m+1)}(x+t)\,dt + \tfrac{1}{2}\int_0^{\infty} \frac{\bar{E}_m^{(1)}(h+\omega_1-t)}{m!}\,\varphi^{(m+1)}(x+t)\,dt + \tfrac{1}{2}\int_0^{\infty} \frac{\bar{E}_m^{(1)}(h-t)}{m!}\,\varphi^{(m+1)}(x+t)\,dt.$$

On aura donc

$$R_m^{(1)}(x) = \int_0^{\infty} \frac{\bar{E}_m^{(1)}(h-t)}{m!} \underset{\omega_1}{\nabla}\, \varphi^{(m+1)}(x+t)\,dt.$$

L'équation (27) est par conséquent vraie si $n = 1$. Supposons que cette équation est vraie si $n = p-1$. Nous avons vu que

$$R_m^{(p)}(x) = Q_m^{(p)} + R_m^{(p-1)}(x).$$

On a donc

$$R_m^{(p)}(x) = -\tfrac{1}{2}\int_0^{\omega_p} \frac{\bar{E}_m^{(p)}(h+\omega_p-t)}{m!} \overset{p-1}{\underset{\omega_1\dots\omega_{p-1}}{\nabla}}\, \varphi^{(m+1)}(x+t)\,dt + \int_0^{\infty} \frac{\bar{E}_m^{(p-1)}(h-t)}{m!} \overset{p-1}{\underset{\omega_1\dots\omega_{p-1}}{\nabla}}\, \varphi^{(m+1)}(x+t)\,dt.$$

Mais puisque

$$\underset{\omega_p}{\nabla}\, \bar{E}_m^{(p)}(x) = \bar{E}_m^{(p-1)}(x),$$

on en déduit

$$R_m^{(p)}(x) = \tfrac{1}{2}\int_{\omega_p}^{\infty} \frac{\bar{E}_m^{(p)}(h-t+\omega_p)}{m!} \overset{p-1}{\underset{\omega_1\dots\omega_{p-1}}{\nabla}}\, \varphi^{(m+1)}(x+t)\,dt + \tfrac{1}{2}\int_0^{\infty} \frac{\bar{E}_m^{(p)}(h-t)}{m!} \overset{p-1}{\underset{\omega_1\dots\omega_{p-1}}{\nabla}}\, \varphi^{(m+1)}(x+t)\,dt.$$

Il en résulte que

$$R_m^{(p)}(x) = \int_0^\infty \frac{\bar{E}_m^{(p)}(h-t)}{m!} \underset{\omega_1 \dots \omega_p}{\overset{p}{\nabla}} \varphi^{(m+1)}(x+t)\, dt, \text{ c. q. f. d.}$$

5. Je vais donner une seconde démonstration de la formule fondamentale (21). Cette démonstration est plus simple que celle qui précède mais elle est en revanche moins générale. Je suppose maintenant que la fonction $\varphi(x)$ et ses dérivées des $m+1$ premiers ordres satisfont aux égalités

$$\lim_{x \to \infty} x^{n+\varepsilon} \varphi^{(\nu)}(x) = 0, \qquad (\nu = 0, 1, 2, \dots, m+1)$$

ε étant un nombre positif. Supposons comme plus haut que

$$0 \leqq h < \omega_1 + \omega_2 + \cdots + \omega_n$$

et considérons l'intégrale (27) qui évidemment a un sens. En intégrant par partie on trouve

$$R_m^{(n)}(x) = R_{m-1}^{(n)}(x) - \frac{E_m^{(n)}(h)}{m!} \underset{\omega_1 \dots \omega_n}{\overset{n}{\nabla}} \varphi^{(m)}(x).$$

En répétant cette opération m fois il vient

$$R_m^{(n)}(x) = R_0^{(n)}(x) - \sum_{\nu=1}^{\nu=m} \frac{E_\nu^{(n)}(h)}{\nu!} \underset{\omega_1 \dots \omega_n}{\overset{n}{\nabla}} \varphi^{(\nu)}(x). \tag{28}$$

Il nous reste à évaluer l'intégrale

$$R_0^{(n)}(x) = \int_0^\infty \bar{E}_0^{(n)}(h-t) \underset{\omega_1 \dots \omega_n}{\overset{n}{\nabla}} \varphi'(x+t)\, dt.$$

La fonction $\bar{E}_0^{(n)}(h-t)$ est discontinue dans les points $t = h + s_1\omega_1 + s_2\omega_2 + \cdots + s_n\omega_n$, où les s_i sont des entiers non négatifs, et elle n'admet pas d'autres points de discontinuité dans l'intervalle d'intégration. Elle est d'ailleurs égale à une constante dans tout intervalle qui ne renferme pas de point de dis-

continuité. En tenant compte de ce que nous avons dit des discontinuités dans le paragraphe 2 on voit que cette intégrale est égale à

$$(29)\quad R_0^{(n)}(x) = -\underset{\omega_1 \dots \omega_n}{\overset{n}{\nabla}} \varphi(x) + 2^n \sum (-1)^{s_1+s_2+\dots+s_n} \underset{\omega_1 \dots \omega_n}{\overset{n}{\nabla}} \varphi(x+h+\Omega)$$

où l'on a posé

$$\Omega = s_1\omega_1 + s_2\omega_2 + \dots + s_n\omega_n.$$

Dans cette série la sommation est étendue à toutes les valeurs entières, non négatives de $s_1, s_2, \dots, s_n$. La série est évidemment absolument convergente parce que

$$\lim_{x\to\infty} x^{n+\varepsilon}\varphi(x) = 0.$$

Pour trouver sa valeur remarquons qu'on a

$$\sum_{s_1=0}^{s_1=p_1-1} \sum_{s_2=0}^{s_2=p_2-1} \cdots \sum_{s_n=0}^{s_n=p_n-1} (-1)^{s_1+s_2+\dots+s_n} \underset{\omega_1 \dots \omega_n}{\overset{n}{\nabla}} \varphi(x+\Omega) = \underset{p_1\omega_1,\dots,p_n\omega_n}{\overset{n}{\nabla}} \varphi(x),$$

$p_1, p_2, \dots, p_n$ étant des entiers positifs impairs quelconques. En faisant tendre ces entiers vers l'infini il vient

$$\sum (-1)^{s_1+s_2+\dots+s_n} \underset{\omega_1 \dots \omega_n}{\overset{n}{\nabla}} \varphi(x+\Omega) = 2^{-n}\varphi(x).$$

L'équation (29) se réduit par conséquent à

$$R_0^{(n)}(x) = -\underset{\omega_1 \dots \omega_n}{\overset{n}{\nabla}} \varphi(x) + \varphi(x+h).$$

En substituant cette expression dans l'équation (28) on trouve la relation suivante

$$\varphi(x+h) = \sum_{\nu=0}^{\nu=m} \frac{E_\nu^{(n)}(h)}{\nu!} \underset{\omega_1 \dots \omega_n}{\overset{n}{\nabla}} \varphi^{(\nu)}(x) + \int_0^\infty \frac{\overline{E}_m^{(n)}(h-t)}{m!} \underset{\omega_1 \dots \omega_n}{\overset{n}{\nabla}} \varphi^{(m+1)}(x+t)\,dt.$$

6. Cette formule peut nous donner une règle de convergence qui n'est pas sans intérêt. Considérons la série à n entrées

$$(30)\quad \sum (-1)^{s_1+s_2+\dots+s_n} \varphi(x+\Omega),$$

la sommation étant étendue à toutes les valeurs entières, non négatives de $s_1, s_2, \dots, s_n$. Supposons que la fonction $\varphi(x)$ tend vers zéro quand x augmente indéfiniment et qu'elle admet, pour $x \geqq a$, une dérivée continue d'un certain ordre, soit d'ordre m, telle que

$$\lim_{x\to\infty} x^{n+\varepsilon}\,\varphi^{(m)}(x) = 0, \qquad (\varepsilon > 0). \tag{31}$$

Je vais démontrer que la série (30) converge uniformément pour $x \geqq a$. En effet, soit h un nombre tel que $0 \leqq h < \omega_1 + \omega_2 + \cdots + \omega_n$. Nous avons vu qu'on a

$$\varphi(x+h) = \sum_{\nu=0}^{\nu=m-1} \frac{E_\nu^{(n)}(h)}{\nu!} \underset{\omega_1 \dots \omega_n}{\overset{n}{\nabla}} \varphi^{(\nu)}(x) + \int_0^\infty \frac{\overline{E}_{m-1}^{(n)}(h-t)}{(m-1)!} \underset{\omega_1 \dots \omega_n}{\overset{n}{\nabla}} \varphi^{(m)}(x+t)\,dt.$$

On en déduit la relation suivante

$$\sum_{s_n=0}^{p_n-1} \cdots \sum_{s_2=0}^{p_2-1} \sum_{s_1=0}^{p_1-1} (-1)^{s_1+s_2+\cdots+s_n} \varphi(x+h+\Omega) \tag{32}$$

$$= \sum_{\nu=0}^{\nu=m-1} \frac{E_\nu^{(n)}(h)}{\nu!} \underset{p_1\omega_1, \dots, p_n\omega_n}{\overset{n}{\nabla}} \varphi^{(\nu)}(x) + \int_0^\infty \frac{\overline{E}_{m-1}^{(n)}(h-t)}{(m-1)!} \underset{p_1\omega_1, \dots, p_n\omega_n}{\overset{n}{\nabla}} \varphi^{(m)}(x+t)\,dt,$$

$p_1, p_2, \dots, p_n$ étant des entiers positifs impairs quelconques. Si un ou plusieurs des nombres p_i sont pairs on trouve une relation analogue, mais un peu plus compliquée, que nous nous dispenserons de reproduire. Il s'agit de démontrer que le second membre de l'équation (32) tend vers une limite quand tous les p_i tendent vers l'infini indépendamment l'un de l'autre. L'intégrale au second membre peut se décomposer en une somme de 2^n intégrales dont l'une est égale à

$$2^{-n} \int_0^\infty \frac{\overline{E}_{m-1}^{(n)}(h-t)}{(m-1)!} \varphi^{(m)}(x+t)\,dt, \tag{33}$$

pendant que les autres sont de la forme

$$P = 2^{-n} \int_0^\infty \frac{\overline{E}_{m-1}^{(n)}(h-t)}{(m-1)!} \varphi^{(m)}(x+p+t)\,dt, \tag{34}$$

p étant un nombre qui tend vers l'infini quand tous les entiers p_i tendent vers l'infini. On aura, pour $t>0$,

$$|\bar{E}_{m-1}^{(n)}(h-t)|<Ct^{n-1},$$

C étant une constante. Il résulte donc de l'équation (31) qu'on sait trouver une constante C_1 telle que

$$|P|<C_1\int_0^\infty \frac{t^{n-1}\,dt}{(x+p+t)^{n+\varepsilon}}<C_1\int_0^\infty \frac{dt}{(x+p+t)^{1+\varepsilon}}=\frac{C_1}{\varepsilon}\,\frac{1}{(x+p)^\varepsilon}.$$

ε étant positif on conclut de cette inégalité que les intégrales (34) tendent uniformément vers zéro. L'intégrale au second membre de l'équation (32) tend donc uniformément vers l'intégrale (33) quand les p_i augmentent indéfiniment.

Cela posé, rappelons le théorème suivant, dû à Hardy et Littlewood*: Si la fonction $f(x)$ tend vers zéro, quand $x\to\infty$, et admet une dérivée d'ordre m, qui est continue et bornée, alors les dérivées d'ordres inférieurs à m tendent nécessairement vers zéro quand $x\to\infty$.

De nos hypothèses relativement à la fonction $\varphi(x)$ il résulte donc que

$$\lim_{x\to\infty}\varphi^{(\nu)}(x)=0, \qquad (\nu=0,1,2,\ldots,m).$$

Le premier terme au second membre de l'équation (32) tend donc uniformément vers une limite quand les p_i augmentent indéfiniment. La série (30) est par conséquent uniformément convergente et l'on aura

$$2^n\sum(-1)^{s_1+s_2+\ldots+s_n}\varphi(x+\Omega)$$
$$=\sum_{\nu=0}^{\nu=m-1}\frac{E_\nu^{(n)}(h)}{\nu!}\varphi^{(\nu)}(x-h)+\int_{-h}^{\infty}\frac{\bar{E}_{m-1}^{(n)}(-t)}{(m-1)!}\varphi^{(m)}(x+t)\,dt.$$

On voit ainsi que la série (30) converge par exemple dans le cas suivant

$$\varphi(x)=\frac{1}{x^\beta(\log x)^\gamma}, \qquad \Re(\beta)>0,$$

et cela quel que soit γ. Mais si $\Re(\beta)=0$, la série converge pourvu que $\Re(\gamma)>0$.

*Proceedings of the London Mathematical Society, ser. 2, vol. 9 (1911), pp. 437–438; ser. 2, vol. 11 (1913), pp. 422–423.

Dans une série multiple on peut effectuer la sommation d'une infinité de manières. Nous avons choisi celle qui est la plus naturelle, mais il convient de remarquer que, si l'on adopte une autre méthode de sommation, les conditions de convergence changent entièrement à moins que la série ne converge absolument. Considérons par exemple la série (30) et supposons pour un moment que tous les nombres ω_i se réduisent à 1. Il paraît naturel de transformer la série en une série simple de la manière suivante. On a évidemment

$$\sum_{s_1+s_2+\cdots+s_n\leq p}(-1)^{s_1+s_2+\cdots+s_n}\varphi(x+s_1+s_2+\cdots+s_n)$$
$$=\sum_{s=0}^{s=p}\frac{(s+1)(s+2)\cdots(s+n-1)}{1\cdot 2\cdot 3\cdots(n-1)}(-1)^s\varphi(x+s).$$

Dans le premier membre la sommation est étendue à toutes les valeurs entières, non négatives de $s_1, s_2, \ldots, s_n$ qui vérifient l'inégalité $s_1+s_2+\cdots+s_n\leq p$. Faisons tendre l'entier p vers l'infini. Il vient

$$\sum_{s_1,\ldots,s_n=0,1,2\ldots}(-1)^{s_1+s_2+\cdots+s_n}\varphi(x+s_1+s_2+\cdots+s_n)$$
$$=\sum_{s=0}^{\infty}\frac{(s+1)(s+2)\cdots(s+n-1)}{1\cdot 2\cdot 3\cdots(n-1)}(-1)^s\varphi(x+s),$$

pourvu que la série au premier membre soit absolument convergente. Mais s'il n'en est pas ainsi cette égalité n'a pas lieu, en général. Soit par exemple

$$\varphi(x)=\frac{1}{x^\beta}.$$

Nous venons de voir que la série multiple au premier membre converge, si $\beta>0$. Mais pour que la série simple au second membre converge il faut que $\beta>n-1$. On peut aussi transformer la série (30) en une série n fois itérée. Admettons comme plus haut que les ω_i soient des nombres positifs quelconques. Reprenons l'équation (32) et faisons tendre d'abord p_1, puis $p_2, \ldots$ et enfin p_n vers l'infini. Le second membre tend vers une limite et vers la même limite que dans le cas où ces nombres tendent simultanément vers l'infini. La série n fois itérée

$$\sum_{s_n=0}^{\infty}\left(\cdots\sum_{s_2=0}^{\infty}\left(\sum_{s_1=0}^{\infty}(-1)^{s_1+s_2+\cdots+s_n}\varphi(x+\Omega)\right)\right)$$

est donc convergente, si $\varphi(x)$ satisfait aux conditions énumérées au commencement de ce paragraphe, et cette série a la même somme que la série multiple (30).

7. Les polynomes d'Euler figurent comme coefficients dans la série suivante*:

$$\frac{e^{-h\eta}\,2^n}{(e^{-\eta\omega_1}+1)(e^{-\eta\omega_2}+1)\dots(e^{-\eta\omega_n}+1)} = \sum_{\nu=0}^{\infty} \frac{E_\nu^{(n)}(h)}{\nu!}(-\eta)^\nu.$$

Si l'on arrête la série au second membre à un certain terme on peut trouver une expression bien simple du terme complémentaire. En effet, posons dans l'équation (21)

$$\varphi(x) = e^{-\eta x},$$

η étant un nombre positif. On aura

$$\underset{\omega_1 \dots \omega_n}{\overset{n}{\nabla}} \varphi^{(\nu)}(x) = 2^{-n}(-\eta)^\nu e^{-\eta x}(e^{-\omega_1\eta}+1)(e^{-\omega_2\eta}+1)\cdots(e^{-\omega_n\eta}+1).$$

En substituant ces expressions dans l'équation (21) on trouvera

$$\frac{e^{-\eta h}\,2^n}{(e^{-\omega_1\eta}+1)\,(e^{-\omega_2\eta}+1)\cdots(e^{-\omega_n\eta}+1)}$$

$$= \sum_{\nu=0}^{\nu=m} \frac{E_\nu^{(n)}(h)}{\nu!}(-\eta)^\nu + (-\eta)^{m+1}\int_0^\infty \frac{\bar{E}_m^{(n)}(h-t)}{m!}\,e^{-\eta t}\,dt.$$

De cette relation on peut déduire un lemme qui nous sera utile plus loin. En développant le premier membre suivant les puissances de η on obtient

$$\int_0^\infty \frac{\bar{E}_m^{(n)}(h-t)}{m!}\,e^{-\eta t}\,dt = \sum_{\nu=0}^{\infty} \frac{E_{\nu+m+1}^{(n)}(h)}{(\nu+m+1)!}(-\eta)^\nu.$$

Dérivons p fois par rapport à η, il vient

$$\int_0^\infty \frac{\bar{E}_m^{(n)}(h-t)}{m!}\,t^p\,e^{-\eta t}\,dt = \sum_{\nu=0}^{\infty} \frac{E_{\nu+m+p+1}^{(n)}(h)}{(\nu+m+p+1)!}\,\frac{(\nu+p)!}{\nu!}(-\eta)^\nu.$$

* Acta Mathematica, vol. 43, p. 184.

L'intégrale au premier membre converge pour $\eta > 0$ et diverge pour $\eta = 0$. Elle représente, pour $\eta > 0$, une fonction analytique de η qui est holomorphe dans le point $\eta = 0$. L'intégrale tend donc vers une limite quand le nombre positif η tend vers zéro et l'on trouve

$$\lim_{\eta \to 0} \int_0^\infty \bar{E}_m^{(n)}(h-t)\, t^p\, e^{-\eta t}\, dt = \frac{m!\, p!}{(m+p+1)!}\, E_{m+p+1}^{(n)}(h).$$

La solution principale $G_n(x)$

8. Ces préliminaires posés je vais démontrer l'existence de la fonction $G_n(x \mid \omega_1, \omega_2, \ldots, \omega_n)$ définie par l'équation (4). Soit $\varphi(x)$ une fonction qui, pour $x \geq a$, admet une dérivée continue d'un certain ordre, soit d'ordre $m+1$, telle que

$$\lim_{x \to \infty} x^{n+\varepsilon} \varphi^{(m+1)}(x) = 0, \tag{35}$$

ε étant un nombre positif. Considérons la série (3). Cette série converge pour toute valeur positive de η car de l'hypothèse que nous venons de faire il résulte qu'on sait trouver une constante c telle que

$$|\varphi^{(\nu)}(x)| < c x^m, \qquad (\nu = 0, 1, 2, \ldots, m), \tag{36}$$

quelque grand que soit x. Cette inégalité est en particulier vraie pour $\nu = 0$. La série (3) converge donc uniformément quand x reste dans un intervalle fini quelconque $a \leq x \leq b$. Faisons tendre le nombre positif η vers zéro. Je veux démontrer que l'expression (3) tend uniformément vers une limite que je désigne par $G_n(x \mid \omega_1, \omega_2, \ldots, \omega_n)$. Reprenons la relation

$$f(x+h) = \sum_{\nu=0}^{\nu=m} \frac{E_\nu^{(n)}(h)}{\nu!} \overset{n}{\underset{\omega_1 \ldots \omega_n}{\nabla}} f^{(\nu)}(x) + \int_0^\infty \frac{\bar{E}_m^{(n)}(h-t)}{m!} \overset{n}{\underset{\omega_1 \ldots \omega_n}{\nabla}} f^{(m+1)}(x+t)\, dt.$$

Soient $p_1, p_2, \ldots, p_n$ des entiers positifs impairs quelconques. Remplaçons x par $x+\Omega$ et donnons aux s_i les valeurs $0, 1, 2, \ldots, p_i - 1$. En ajoutant ensemble toutes les équations ainsi obtenues on trouvera

$$\sum_{s_n=0}^{p_n-1} \cdots \sum_{s_2=0}^{p_2-1} \sum_{s_1=0}^{p_1-1} (-1)^{s_1+s_2+\cdots+s_n} f(x+h+\Omega) \tag{37}$$

$$= \sum_{\nu=0}^{\nu=m} \frac{E_\nu^{(n)}(h)}{\nu!} \overset{n}{\underset{p_1\omega_1, \ldots, p_n\omega_n}{\nabla}} f^{(\nu)}(x) + \int_0^\infty \frac{\bar{E}_m^{(n)}(h-t)}{m!} \overset{n}{\underset{p_1\omega_1, \ldots, p_n\omega_n}{\nabla}} f^{(m+1)}(x+t)\, dt.$$

Posons $f(x) = \varphi(x)\, e^{-\eta x}$ et faisons tendre les entiers $p_1, p_2, \ldots, p_n$ vers l'infini. Il vient

$$(38) \qquad 2^n \sum (-1)^{s_1+s_2+\cdots+s_n} \varphi(x+h+\Omega)\, e^{-\eta(x+h+\Omega)}$$

$$= \sum_{\nu=0}^{\nu=m} \frac{E_\nu^{(m)}(h)}{\nu!} D_x^\nu [\varphi(x)\, e^{-\eta x}] + \int_0^\infty \frac{\bar{E}_m^{(m)}(h-t)}{m!} D_x^{m+1} [\varphi(x+t)\, e^{-\eta(x+t)}]\, dt.$$

On le voit aisément en tenant compte de l'inégalité (36). En effet le premier terme au second membre de l'équation (37) est composé d'un nombre fini de termes de la forme

$$C\, \varphi^{(\nu)}(x+\Omega)\, e^{-\eta(x+\Omega)},$$

C étant une constante. Et ces termes tendent vers zéro quand Ω augmente indéfiniment. L'intégrale au second membre de l'équation (37) peut se décomposer en un nombre fini de termes de la forme

$$Q = C \int_0^\infty \bar{E}_m^{(m)}(h-t)\, \varphi^{(\nu)}(x+\Omega+t)\, e^{-\eta(x+\Omega+t)}\, dt.$$

Mais on aura

$$|Q| < C_1 \int_0^\infty t^{n-1} (x+\Omega+t)^m\, e^{-\eta(x+\Omega+t)}\, dt < C_1 \int_{x+\Omega}^\infty t^{m+n-1}\, e^{-\eta t}\, dt$$

et la dernière intégrale tend vers zéro quand Ω tend vers l'infini, η étant un nombre positif fixe. Cela posé, faisons tendre η vers zéro dans l'équation (38). On trouvera

$$(39)\ G_n(x+h) = \sum_{\nu=0}^{\nu=m} \frac{E_\nu^{(m)}(h)}{\nu!} \varphi^{(\nu)}(x) + \int_0^\infty \frac{\bar{E}_m^{(m)}(h-t)}{m!} \varphi^{(m+1)}(x+t)\, dt.$$

En effet on aura

$$\lim_{\eta \to 0} \int_0^\infty \bar{E}_m^{(m)}(h-t)\, \varphi^{(m+1)}(x+t)\, e^{-\eta(x+t)}\, dt = \int_0^\infty \bar{E}_m^{(m)}(h-t)\, \varphi^{(m+1)}(x+t)\, dt$$

et cela uniformément dans l'intervalle $a \leqq x \leqq b$ parce que l'intégrale au second membre est convergente. Posons

$$P_\nu = \eta^\nu \int_0^\infty \bar{E}_m^{(n)}(h-t)\,\varphi^{(m+1-\nu)}(x+t)\,e^{-\eta(x+t)}\,dt.$$

L'équation (39) est établie si l'on peut démontrer que

$$\lim_{\eta \to 0} P_\nu = 0, \qquad (\nu = 1, 2, 3, \ldots, m+1).$$

Il résulte de l'équation (35) que les dérivées de $\varphi(x)$ sont de la forme

$$\varphi^{(m+1-\nu)}(x) = \psi_\nu(x) + p_\nu(x), \quad (\nu = 1, 2, \ldots, m+1),$$

$p_\nu(x)$ étant un polynome de degré $\nu - 1$ et $\psi_\nu(x)$ étant une fonction continue de x telle que

$$\lim_{x \to \infty} x^{n+\varepsilon-\nu}\,\psi_\nu(x) = 0,$$

ε étant un nombre positif. On a donc

$$P_\nu = \eta^\nu \int_0^\infty \bar{E}_m^{(n)}(h-t)\,\psi_\nu(x+t)\,e^{-\eta(x+t)}\,dt + \eta^\nu \int_0^\infty \bar{E}_m^{(n)}(h-t)\,p_\nu(x+t)\,e^{-\eta(x+t)}\,dt.$$

Dans le paragraphe 7 nous avons démontré que l'intégrale

$$\int_0^\infty \bar{E}_m^{(n)}(h-t)\,t^p\,e^{-\eta t}\,dt$$

tend vers une limite finie quand η tend vers zéro. Comme $p_\nu(x)$ est un polynome le dernier terme au second membre tendra uniformément vers zéro quand η tend vers zéro. Considérons l'autre terme. On sait trouver une constante C telle que

$$\left|\eta^{\nu}\int_0^{\infty}\bar{E}_m^{(n)}(h-t)\,\psi_{\nu}(x+t)\,e^{-\eta(x+t)}\,dt\right|$$

$$<C\eta^{\nu}\int_x^{\infty}t^{n-1}\,t^{\nu-n-\varepsilon}\,e^{-\eta t}\,dt<C\eta^{\varepsilon}\int_0^{\infty}t^{\nu-1-\varepsilon}\,e^{-t}\,dt.$$

De cette inégalité on conclut que P_{ν} tend uniformément vers zéro avec η. Nous avons ainsi démontré que le second membre de l'équation (4) tend vers une limite uniformément dans l'intervalle $a \leqq x \leqq b$. Cette limite est donc une fonction continue de x qui admet un développement de la forme (39). On vérifie immédiatement que la fonction $G_n(x)$ est une solution de l'équation (2). En effet en appliquant l'opération $\bigtriangledown_{\omega_n}$ aux deux membres de l'équation (4) on trouvera

$$\bigtriangledown_{\omega_n} G_n(x \mid \omega_1, \omega_2, \ldots, \omega_n)$$

$$=\lim_{\eta\to 0}2^{n-1}\sum(-1)^{s_1+\cdots+s_{n-1}}\varphi(x+s_1\omega_1+\cdots+s_{n-1}\omega_{n-1})\,e^{-\eta(x+s_1\omega_1+\cdots+s_{n-1}\omega_{n-1})},$$

c'est-à-dire que

$$\bigtriangledown_{\omega_n} G_n(x \mid \omega_1, \ldots, \omega_n) = G_{n-1}(x \mid \omega_1, \ldots, \omega_{n-1}).$$

On en conclut que

$$\bigtriangledown^{2}_{\omega_2\omega_{n-1}} G_n(x \mid \omega_1, \ldots, \omega_n) = G_{n-2}(x \mid \omega_1, \ldots, \omega_{n-2}),$$

. . . . . .

$$\bigtriangledown^{n-1}_{\omega_2\ldots\omega_n} G_n(x \mid \omega_1, \ldots, \omega_n) = G_1(x \mid \omega_1),$$

et puisque

$$\bigtriangledown_{\omega_1} G_1(x \mid \omega_1) = \varphi(x)$$

on trouvera enfin

$$\bigtriangledown^{n}_{\omega_1\ldots\omega_n} G_n(x \mid \omega_1, \ldots, \omega_n) = \varphi(x).$$

II.

Extension de la formule sommatoire d'Euler et de Maclaurin

9. Je vais maintenant établir une certaine identité qui nous sera utile dans l'étude des solutions de l'équation (1). Soit $B_\nu(x)$ le polynome de Bernoulli, c'est-à-dire le polynome qui satisfait à l'équation

$$B_\nu(x+1)-B_\nu(x)=\nu x^{\nu-1}.$$

Soit $\bar{B}_\nu(x)$ une fonction périodique avec la période *un* qui, dans l'intervalle $0\leqq x<1$, est égale au polynome de Bernoulli $B_\nu(x)$. Soit h un nombre tel que $0\leqq h<1$. La formule sommatoire d'Euler et de Maclaurin peut s'écrire comme il suit

$$\varphi(x+h)=\int_x^{x+1}\varphi(t)\,dt+\sum_{\nu=1}^{\nu=m}\frac{B_\nu(h)}{\nu!}\,\triangle\,\varphi^{(\nu-1)}(x)+R_m \tag{1}$$

où

$$R_m=-\int_0^1\frac{\bar{B}_m(h+t)}{m!}\,\varphi^{(m)}(x+1-t)\,dt. \tag{2}$$

On sait que cette relation est très utile quand il s'agit d'évaluer certaines séries à entrée simple. Je vais déduire une relation analogue mais plus générale qui peut servir à évaluer certaines séries multiples. Soit $B_\nu^{(n)}(x)$ le polynome de Bernoulli d'ordre n, on aura*

$$\varphi^{(n)}(x+h)=\sum_{\nu=0}^{\nu=m}\frac{B_\nu^{(n)}(h)}{\nu!}\,\underset{\omega_1\ldots\omega_n}{\triangle}\,\varphi^{(\nu)}(x),$$

$\varphi(x)$ étant un polynome quelconque du degré $m+n$ et $\varphi^{(\nu)}(x)$ étant sa dérivée d'ordre ν. En changeant un peu les notations cette identité peut s'écrire

$$\varphi(x+h)=\sum_{\nu=0}^{\nu=m}\frac{B_\nu^{(n)}(h)}{\nu!}\int_0^1 dt_1\int_0^1 dt_2\cdots\int_0^1\varphi^{(\nu)}(x+\omega_1t_1+\omega_2t_2+\cdots+\omega_nt_n)\,dt_n.$$

* Acta Mathematica, vol. 43, p. 163.

Il va sans dire que ces équations cessent d'être vraies si la fonction $\varphi(x)$ n'est pas un polynome. Et si l'on faisait tendre m vers l'infini on trouvera dans la plupart des cas une série divergente. Mais je vais démontrer que le terme complémentaire de cette série s'exprime aisément par les dérivées de $\varphi(x)$ sous une forme analogue à l'expression (2).

Dans ce but je définis une fonction $\overline{B}_\nu^{(n)}(x \mid \omega_1, \ldots, \omega_n)$ comme la solution de l'équation

$$\overset{n}{\underset{\omega_1 \cdots \omega_n}{\triangle}} \overline{B}_\nu^{(n)}(x) = 0 \tag{3}$$

qui, dans l'intervalle $0 \leqq x < \omega_1 + \omega_2 + \cdots + \omega_n$, est égale au polynome de Bernoulli $B_\nu^{(n)}(x \mid \omega_1, \ldots, \omega_n)$:

$$\overline{B}_\nu^{(n)}(x) = B_\nu^{(n)}(x), \qquad (0 \leqq x < \omega_1 + \omega_2 + \cdots + \omega_n). \tag{4}$$

D'abord la fonction est uniquement définie, pour toutes les valeurs réelles de x par les deux conditions que nous venons de lui imposer. Nous distinguerons trois cas suivant que $\nu \gtreqless n$. Si $\nu < n$ la fonction se réduit au polynome de Bernoulli car ce polynome satisfait à l'équation (3). En ce cas l'équation (4) subsiste donc pour toutes les valeurs de x. Si $\nu > n$ on voit d'abord que notre fonction est continue dans le point $x = \omega_1 + \omega_2 + \cdots + \omega_n$. En effet elle est définie dans ce point par l'équation (3), où l'on pose $x = 0$. Le polynome de Bernoulli satisfait à l'équation

$$\overset{n}{\underset{\omega_1 \cdots \omega_n}{\triangle}} B_\nu^{(n)}(x) = \nu(\nu-1)\cdots(\nu-n+1)x^{\nu-n} \tag{5}$$

et le second membre de cette équation tend vers zéro avec x. Notre fonction prend donc la même valeur que le polynome de Bernoulli dans le point en question. Elle est par conséquent continue. Cela posé, faisons varier x, dans l'équation (3), en partant du point $x = 0$. On obtient successivement, pour toutes les valeurs réelles de x, la valeur de la fonction $\overline{B}_\nu^{(n)}(x)$ comme une somme de $2^n - 1$ fonctions continues. Notre fonction est par conséquent continue partout si $\nu > n$. Mais cela n'est plus vrai si $\nu = n$ parce que le second membre de l'équation (5) se réduit à $n!$. Il en résulte que la fonction $\overline{B}_n^{(n)}(x)$ est discontinue dans le point $x = 0$ et par conséquent dans une infinité d'autres points que nous énumérons plus loin.

Des propriétés du polynome de Bernoulli il résulte que la fonction $\overline{B}_\nu^{(n)}(x)$ satisfait aux trois relations suivantes

$$\bar{B}_\nu^{(n)}(\omega_1+\omega_2+\cdots+\omega_n-x)=(-1)^\nu\bar{B}_\nu^{(n)}(x), \qquad (\nu \gtrless n), \tag{6}$$

$$\underset{\omega_n}{\triangle}\bar{B}_\nu^{(n)}(x)=\nu\bar{B}_{\nu-1}^{(n-1)}(x\,|\,\omega_1,\omega_2,\ldots,\omega_{n-1}), \tag{7}$$

$$\frac{d\bar{B}_\nu^{(n)}(x)}{dx}=\nu\bar{B}_{\nu-1}^{(n)}(x), \qquad (\nu \gtrless n). \tag{8}$$

On le démontre par un raisonnement presque identique à celui du paragraphe 2. Par exemple, la dernière équation est par définition vraie dans l'intervalle $0<x<\omega_1+\omega_2+\cdots+\omega_n$ et les deux membres ont leur différence $n^{\text{ième}}$ égale à nul. Elle est donc vraie pour toutes les valeurs de x. Si $\nu=n$ la dérivée cesse d'exister dans une infinité de points. $\bar{B}_\nu^{(n)}(x)$ est donc, si $\nu>n$, une fonction continue de x qui admet des dérivées continues des ordres $1,2,\ldots,\nu-n-1$. Elle admet en outre une dérivée d'ordre $\nu-n$ qui est discontinue dans une infinité de points.

Considérons en particulier le cas $\nu=n$. La fonction $\bar{B}_1^{(1)}(x\,|\,\omega_1)$ est périodique avec la période ω_1. Elle est toujours continue, excepté pour les valeurs de x de la forme $p\omega_1$ où elle fait un saut brusque égal à $-\omega_1$, c'est-à-dire qu'on a

$$\bar{B}_1^{(1)}(p\omega_1+0)-\bar{B}_1^{(1)}(p\omega_1-0)=-\omega_1,$$

p étant un entier positif, négatif ou nul.

Cela posé, considérons la fonction $\bar{B}_2^{(2)}(x\,|\,\omega_1,\omega_2)$. Elle est, par définition, continue dans l'intervalle $0<x<\omega_1+\omega_2$ et l'on aura

$$\bar{B}_2^{(2)}(x+\omega_2)-\bar{B}_2^{(2)}(x)=2\omega_2\bar{B}_1^{(1)}(x\,|\,\omega_1).$$

Il en résulte que la fonction $\bar{B}_2^{(2)}(x)$ est discontinue dans les points $p_1\omega_1+p_2\omega_2$, p_1 et p_2 étant des entiers positifs, et elle y fait un saut brusque égal à $-2\omega_1\omega_2$. En général, elle est continue dans les points $p_1\omega_1$ et dans les points $p_2\omega_2$. De plus la fonction est discontinue dans les points $-p_1\omega_1-p_2\omega_2$, p_1 et p_2 étant des entiers non négatifs, et elle y fait un saut brusque égal à $2\omega_1\omega_2$. Dans tout autre point la fonction est continue.

A l'aide de la relation

$$\bar{B}_n^{(n)}(x+\omega_n)-\bar{B}_n^{(n)}(x)=n\omega_n\bar{B}_{n-1}^{(n-1)}(x)$$

on démontre par voie d'induction que la fonction $\bar{B}_n^{(n)}(x \mid \omega_1, \omega_2, \ldots, \omega_n)$ est discontinue dans les points $p_1\omega_1 + p_2\omega_2 + \cdots + p_n\omega_n$ de telle manière que

$$\bar{B}_n^{(n)}(p_1\omega_1 + p_2\omega_2 + \cdots + p_n\omega_n + 0) - \bar{B}_n^{(n)}(p_1\omega_1 + p_2\omega_2 + \cdots + p_n\omega_n - 0) = -n!\,\omega_1\omega_2\cdots\omega_n,$$

$p_1, p_2, \ldots, p_n$ étant des entiers positifs. Et elle est discontinue dans les points $-p_1\omega_1 - p_2\omega_2 - \cdots - p_n\omega_n$ de telle manière que

$$\bar{B}_n^{(n)}(-p_1\omega_1 - p_2\omega_2 - \cdots - p_n\omega_n + 0) - \bar{B}_n^{(n)}(-p_1\omega_1 - p_2\omega_2 - \cdots - p_n\omega_n - 0) = (-1)^n n!\,\omega_1\omega_2\cdots\omega_n,$$

$p_1, p_2, \ldots, p_n$ étant des entiers non négatifs. Dans tout autre point la fonction est continue.

S'il arrive que q des points susdits viennent coïncider dans un point nous comptons ce point comme un point de discontinuité q^{iple} et dans un tel point le saut est q fois plus grand que dans un point simple.

Considérons en particulier le cas où tous les nombres $\omega_1, \omega_2, \ldots, \omega_n$ se réduisent à ω. En ce cas on peut donner une expression remarquable de notre fonction. Je dis qu'on a

$$(9)\quad \bar{B}_\nu^{(n+1)}(x) = \frac{\nu!}{n!(\nu-n-1)!}\sum_{s=0}^{s=\nu}\frac{(-1)^s}{s!}\frac{\bar{B}_{\nu-n+s}(x)}{\nu-n+s}D_x^s(x-1)(x-2)\cdots(x-n).$$

En effet, en appliquant l'opération $\triangle$ aux deux membres de cette équation il vient

$$\triangle\,\bar{B}_\nu^{(n+1)}(x) = \nu\,\bar{B}_{\nu-1}^{(n)}(x).$$

D'autre part des propriétés des polynomes de Bernoulli* il résulte que l'équation (9) est vraie dans l'intervalle $0 \leqq x < 1$. Par voie d'induction on démontre alors qu'elle est vraie pour toutes les valeurs de x.

De l'équation (9) il résulte en particulier que

$$\bar{B}_n^{(n)}(x) = n(x-1)(x-2)\cdots(x-n+1)\bar{B}_1(x) + f(x),$$

* Acta Mathematica, vol. 43, p. 188.

$f(x)$ étant une fonction continue. On voit donc que la fonction $\bar{B}_n^{(n)}(x)$ est discontinue dans les points $x = n, n+1, n+2, \ldots$ et dans les points $x = 0, -1, -2, -3, \ldots$ et il n'y a pas d'autres points de discontinuité. Si p est un entier positif, nul ou négatif on aura

$$\bar{B}_n^{(n)}(p+0) - \bar{B}_n^{(n)}(p-0) = -n(p-1)(p-2)\cdots(p-n+1).$$

Ce résultat est en accord avec celui que nous venons de trouver dans le cas général.

Il est facile de trouver une limite supérieure du module de la fonction $\bar{B}_\nu^{(n)}(x \mid \omega_1, \omega_2, \ldots, \omega_n)$. Si $n = 1$ la fonction est périodique et bornée. Si $n = 2$ on aura

$$\bar{B}_\nu^{(2)}(x + p\,\omega_2) = \bar{B}_\nu^{(2)}(x) + \omega_2 \nu \sum_{s=0}^{p-1} \bar{B}_{\nu-1}^{(1)}(x + s\,\omega_2 \mid \omega_1).$$

La fonction

$$\frac{\bar{B}_\nu^{(2)}(x)}{x}$$

reste donc comprise entre des limites finies quand $|x|$ tend vers l'infini. Par voie d'induction on démontre qu'on sait trouver une constante positive K telle que

$$|\bar{B}_\nu^{(n)}(x \mid \omega_1, \ldots, \omega_n)| < K|x|^{n-1}$$

quelque grand que soit $|x|$. On peut vérifier l'exactitude de ce résultat sur l'équation (9). Il résulte, en effet, de cette équation que $\bar{B}_\nu^{(n)}(x)$ est de la forme

$$\frac{\bar{B}_\nu^{(n)}(x)}{x^{n-1}} = \pi(x) + \varepsilon(x),$$

$\pi(x)$ étant une fonction périodique et bornée, qui n'est pas identiquement zéro, $\varepsilon(x)$ étant une fonction qui tend vers zéro quand $|x|$ tend vers l'infini.

10. Cela posé, soient m et n deux entiers positifs et supposons que la fonction $\varphi(z)$ admet une dérivée continue d'ordre $m+1$ dans l'intervalle $x \leqq z \leqq x + \omega_1 + \omega_2 + \cdots + \omega_n$. Considérons l'intégrale

$$(10)\quad Q_{m+1}^{(n)} = -\frac{1}{\omega_n}\int_0^{\omega_n} \frac{\overline{B}_{m+n}^{(n)}(h+t)}{(m+n)!} \underset{\omega_1 \dots \omega_{n-1}}{\overset{n-1}{\triangle}} \varphi^{(m+1)}(x+\omega_n - t)\,dt,$$

h étant un nombre tel que

$$(11)\quad 0 \leqq h < \omega_1 + \omega_2 + \cdots + \omega_n.$$

En intégrant par partie on trouve

$$Q_{m+1}^{(n)} = Q_m^{(n)} + \frac{1}{\omega_n}\frac{\overline{B}_{m+n}^{(n)}(h+\omega_n)}{(m+n)!} \underset{\omega_1 \dots \omega_{n-1}}{\overset{n-1}{\triangle}} \varphi^{(m)}(x) - \frac{1}{\omega_n}\frac{\overline{B}_{m+n}^{(n)}(h)}{(m+n)!} \underset{\omega_1 \dots \omega_{n-1}}{\overset{n-1}{\triangle}} \varphi^{(m)}(x+\omega_n).$$

Mais en tenant compte des propriétés de la fonction $\overline{B}(x)$ on peut écrire cette relation comme il suit:

$$(12)\quad Q_{m+1}^{(n)} = Q_m^{(n)} - \frac{\overline{B}_{m+n}^{(n)}(h)}{(m+n)!} \underset{\omega_1 \dots \omega_n}{\overset{n}{\triangle}} \varphi^{(m)}(x) + \frac{\overline{B}_{m+n-1}^{(n-1)}(h)}{(m+n-1)!} \underset{\omega_1 \dots \omega_{n-1}}{\overset{n-1}{\triangle}} \varphi^{(m)}(x).$$

Si $n = 1$ le dernier terme au second membre s'annule parce que

$$\overline{B}_m^{(1)}(h+\omega_1) = \overline{B}_m^{(1)}(h).$$

Je dis maintenant que le terme complémentaire de la formule que j'ai en vue est de la forme

$$(13)\quad R_m^{(n)}(x) = Q_m^{(n)} + Q_m^{(n-1)} + \cdots + Q_m^{(1)}.$$

En effet, de l'équation (12) on déduit d'abord que

$$(14)\quad R_{m+1}^{(n)}(x) = R_m^{(n)}(x) - \frac{B_{m+n}^{(n)}(h)}{(m+n)!} \underset{\omega_1 \dots \omega_n}{\overset{n}{\triangle}} \varphi^{(m)}(x).$$

Par conséquent on aura

$$R_{m+1}^{(n)}(x) = R_1^{(n)}(x) - \sum_{\nu=1}^{\nu=m} \frac{B_{\nu+n}^{(n)}(h)}{(\nu+n)!} \underset{\omega_1 \dots \omega_n}{\overset{n}{\triangle}} \varphi^{(\nu)}(x). \tag{15}$$

Il nous reste à évaluer $R_1^{(n)}(x)$. Dans ce but supposons pour un moment que $0 \leqq h < \omega_1$. L'équation (12) est encore vraie si $m = 0$ et $n > 1$. Mais ce n'est plus le cas, si $m = 0$ et $n = 1$, car la fonction $\bar{B}_1^{(1)}(h+t)$ est discontinue dans le point $t = \omega_1 - h$ qui est situé dans l'intervalle d'intégration. En tenant compte de ce fait on trouve au lieu de (12) l'équation suivante

$$Q_1^{(1)} = Q_0^{(1)} - B_1^{(1)}(h) \underset{\omega_1}{\triangle} \varphi(x) + \varphi(x+h). \tag{16}$$

Par conséquent, si $m = 0$, l'équation (14) doit être remplacée pa la suivante

$$R_1^{(n)}(x) = R_0^{(n)}(x) - \frac{B_n^{(n)}(h)}{n!} \underset{\omega_1 \dots \omega_n}{\overset{n}{\triangle}} \varphi(x) + \varphi(x+h). \tag{17}$$

Nous venons de supposer que $0 \leqq h < \omega_1$. Faisons maintenant croître h mais de sorte qu'il reste toujours dans l'intervalle (11). Alors les fonctions $\bar{B}_s^{(s)}(h+t)$ $(s = 1, 2, \dots, n)$ deviennent discontinues dans un nombre fini de points situés dans les intervalles d'intégration. Il en résulte qu'on doit modifier les équations (12) et (16), si $m = 0$. Dans les seconds membres de ces équations il surgit un nombre fini de termes provenant de ces discontinuités. Mais en se reportant à ce que nous avons dit dans le paragraphe précédent on vérifie sans trop de peine que ces termes se détruisent mutuellement de sorte que l'équation (17) reste vraie quel que soit h dans l'intervalle (11). En intégrant n fois par partie dans l'expression $R_0^{(n)}(x)$ on trouvera enfin

$$R_0^{(n)}(x) = -\sum_{\nu=0}^{\nu=n-1} \frac{B_\nu^{(n)}(h)}{\nu!} \int_0^1 dt_1 \int_0^1 dt_2 \cdots \int_0^1 \varphi^{(\nu)}(x + \omega_1 t_1 + \cdots + \omega_n t_n)\, dt_n.$$

En rapprochant cette équation aux équations (17) et (15) on obtiendra la relation suivante

$$(18)\quad \varphi(x+h)=\sum_{\nu=0}^{\nu=n-1}\frac{B_\nu^{(n)}(h)}{\nu!}\int_0^1 dt_1\int_0^1 dt_2\cdots\int_0^1\varphi^{(\nu)}(x+\omega_1 t_1+\cdots+\omega_n t_n)\,dt_n$$

$$+\sum_{\nu=0}^{\nu=m}\frac{B_{\nu+n}^{(n)}(h)}{(\nu+n)!}\overset{n}{\underset{\omega_1\cdots\omega_n}{\triangle}}\varphi^{(\nu)}(x)+R_{m+1}^{(n)}(x),$$

où l'on a posé

$$(19)\quad R_{m+1}^{(n)}(x)=-\sum_{p=1}^{p=n}\frac{1}{\omega_p}\int_0^{\omega_p}\frac{\bar{B}_{m+p}^{(p)}(h+t)}{(m+p)!}\overset{p-1}{\underset{\omega_1\cdots\omega_{p-1}}{\triangle}}\varphi^{(m+1)}(x+\omega_p-t)\,dt.$$

C'est la formule que j'avais en vue; pour $n=1$ elle se réduit à la formule sommatoire d'Euler. Si l'on fait tendre h vers zéro et si l'on remplace en même temps $\varphi(x)$ par $f^{(n)}(x)$ notre relation prend la forme*

$$(20)\quad f^{(n)}(x)=\sum_{\nu=0}^{\nu=m-1}\frac{B_\nu^{(n)}}{\nu!}\overset{n}{\underset{\omega_1\cdots\omega_p}{\triangle}}f^{(\nu)}(x)$$

$$-\sum_{p=0}^{p=n-1}\frac{1}{\omega_{p+1}}\int_0^{\omega_{p+1}}\frac{B_{m-n+p}^{(p+1)}(t)}{(m-n+p)!}\overset{p}{\underset{\omega_1\cdots\omega_p}{\triangle}}f^{(m)}(x+\omega_{p+1}-t)\,dt,$$

m étant un entier plus grand que n.

11. Chaque terme dans le développement (18) est une fonction symétrique de $\omega_1, \omega_2, \ldots, \omega_n$, le terme complémentaire sera donc aussi une fonction symétrique de ces variables mais cette propriété n'apparaît pas dans l'expression (19). Je vais donner une seconde expression du reste où la symétrie est mise en évidence. Posons

$$P_{m+1}^{(n)}(z)=-\int_0^z\frac{\bar{B}_{m+n}^{(n)}(h+z-t)}{(m+n)!}\,\varphi^{(m+1)}(x+t)\,dt.$$

* Nous avons supposé que les ω_i étaient des nombres positifs. Mais on peut ici faire la même remarque que dans la note de page 17. L'équation (20) reste vraie si les ω_i sont des nombres complexes quelconques et la démonstration précédente s'applique sans modification.

En intégrant par partie on trouvera

$$P_{m+1}^{(n)}(z) = P_m^{(n)}(z) - \frac{B_{m+n}^{(n)}(h)}{(m+n)!}\varphi^{(m)}(x+z) + \frac{\bar{B}_{m+n}^{(n)}(h+z)}{(m+n)!}\varphi^{(m)}(x).$$

Appliquons aux deux membres de cette équation l'opération $\overset{n}{\triangle}$ où $\triangle$ porte sur la variable z. Il vient

$$\overset{n}{\underset{\omega_1 \dots \omega_n}{\triangle}} P_{m+1}^{(n)}(z) = \overset{n}{\underset{\omega_1 \dots \omega_n}{\triangle}} P_m^{(n)}(z) - \frac{B_{m+n}^{(n)}(h)}{(m+n)!} \overset{n}{\underset{\omega_1 \dots \omega_n}{\triangle}} \varphi^{(m)}(x+z).$$

En faisant tendre z vers zéro et en raisonnant comme plus haut, on voit que le terme complémentaire de la série (18) peut s'écrire sous la forme

$$R_m^{(n)}(x) = \left[\overset{n}{\underset{\omega_1 \dots \omega_n}{\triangle}} P_m^{(n)}(z) \right]_{z=0} \tag{21}$$

où l'on a posé $z = 0$ dans la différence d'ordre n.

Dans certaines applications de la série (18) il paraît plus avantageux de se servir d'une d'autre expression du terme complémentaire; mais cette expression demande une nouvelle hypothèse relativement à la fonction $\varphi(z)$. Admettons que la dérivée $\varphi^{(m+1)}(z)$ est continue pour $z \geqq x$, et que l'intégrale

$$\int_0^\infty \bar{B}_{m+n}^{(n)}(-t)\,\varphi^{(m+1)}(z+t)\,dt \tag{22}$$

converge.* Dans ces conditions on aura

$$R_{m+1}^{(n)}(x) = \int_0^\infty \frac{\bar{B}_{m+n}^{(n)}(h-t)}{(m+n)!} \overset{n}{\underset{\omega_1 \dots \omega_n}{\triangle}} \varphi^{(m+1)}(x+t)\,dt. \tag{23}$$

---

* L'intégrale (22) est en particulier absolument convergente si l'on a

$$\lim_{x \to \infty} x^{m+\varepsilon}\varphi^{(m+1)}(x) = 0, \qquad \varepsilon > 0.$$

On le démontre aisément par voie d'induction. En effet de la convergence de l'intégrale (22) il résulte d'abord que toutes les intégrales

$$\int_0^\infty \bar{B}^{(p)}_{m+p}(-t)\,\varphi^{(m+1)}(z+t)\,dt, \qquad (p=1,2\ldots,n)$$

convergent. Si $n=1$ l'équation (19) se réduit à

$$R^{(1)}_{m+1}(x) = -\frac{1}{\omega_1}\int_0^{\omega_1} \frac{\bar{B}^{(1)}_{m+1}(h+\omega_1-t)}{(m+1)!}\,\varphi^{(m+1)}(x+t)\,dt.$$

Mais comme la fonction $\bar{B}^{(1)}_{m+1}(x)$ admet la période ω_1 cette équation peut s'écrire:

$$R^{(1)}_{m+1}(x) = -\frac{1}{\omega_1}\int_0^{\omega_1} \frac{\bar{B}^{(1)}_{m+1}(h+\omega_1-t)}{(m+1)!}\,\varphi^{(m+1)}(x+t)\,dt$$

$$+\frac{1}{\omega_1}\int_0^{\infty} \frac{\bar{B}^{(1)}_{m+1}(h+\omega_1-t)-\bar{B}^{(1)}_{m+1}(h-t)}{(m+1)!}\,\varphi^{(m+1)}(x+t)\,dt.$$

Par conséquent

$$R^{(1)}_{m+1}(x) = \frac{1}{\omega_1}\int_0^{\infty} \frac{\bar{B}^{(1)}_{m+1}(h-t)}{(m+1)!}\,\varphi^{(m+1)}(x+\omega_1+t)\,dt$$

$$-\frac{1}{\omega_1}\int_0^{\infty} \frac{\bar{B}^{(1)}_{m+1}(h-t)}{(m+1)!}\,\varphi^{(m+1)}(x+t)\,dt = \int_0^{\infty} \frac{\bar{B}^{(1)}_{m+1}(h-t)}{(m+1)!}\,\underset{\omega_1}{\triangle}\,\varphi^{(m+1)}(x+t)\,dt.$$

L'équation (23) est donc vraie si $n=1$. Supposons que l'équation (23) soit vraie si $n=p-1$. De la formule (19) il résulte que

$$R^{(p)}_{m+1}(x) = R^{(p-1)}_{m+1}(x) + Q^{(p)}_{m+1}.$$

On aura donc

$$R_{m+1}^{(p)}(x) = \int_0^\infty \frac{\bar{B}_{m+p-1}^{(p-1)}(h-t)}{(m+p-1)!} \underset{\omega_1 \dots \omega_{p-1}}{\overset{p-1}{\triangle}} \varphi^{(m+1)}(x+t)\,dt$$

$$- \frac{1}{\omega_p} \int_0^{\omega_p} \frac{\bar{B}_{m+p}^{(p)}(h+\omega_p-t)}{(m+p)!} \underset{\omega_1 \dots \omega_{p-1}}{\overset{p-1}{\triangle}} \varphi^{(m+1)}(x+t)\,dt.$$

Dans la première intégrale je remplace $\bar{B}_{m+p-1}^{(p-1)}(x)$ par

$$\frac{\bar{B}_{m+p}^{(p)}(x+\omega_p) - \bar{B}_{m+p}^{(p)}(x)}{\omega_p(m+p)}.$$

Il vient

$$R_{m+1}^{(p)}(x) = \frac{1}{\omega_p} \int_{\omega_p}^\infty \frac{\bar{B}_{m+p}^{(p)}(h+\omega_p-t)}{(m+p)!} \underset{\omega_1 \dots \omega_{p-1}}{\overset{p-1}{\triangle}} \varphi^{(m+1)}(x+t)\,dt$$

$$- \frac{1}{\omega_p} \int_0^\infty \frac{\bar{B}_{m+p}^{(p)}(h-t)}{(m+p)!} \underset{\omega_1 \dots \omega_{p-1}}{\overset{p-1}{\triangle}} \varphi^{(m+1)}(x+t)\,dt.$$

Il en résulte que

$$R_{m+1}^{(p)}(x) = \int_0^\infty \frac{\bar{B}_{m+p}^{(p)}(h-t)}{(m+p)!} \underset{\omega_1 \dots \omega_p}{\overset{p}{\triangle}} \varphi^{(m+1)}(x+t)\,dt, \qquad \text{c.q.f.d.}$$

12. Je vais donner une autre démonstration de l'équation (18) qui est plus directe que celle qui précède mais qui demande des hypothèses nouvelles. Je suppose que la fonction $\varphi(x)$ tend vers zéro quand x croît indéfiniment et que ses dérivées tendent vers zéro de telle manière que

$$\lim_{x \to \infty} x^{n+\varepsilon} \underset{\omega_1 \dots \omega_n}{\overset{n}{\triangle}} \varphi^{(\nu)}(x) = 0, \qquad (\nu = 0, 1, 2, \dots, m+1),$$

ε étant un nombre positif aussi petit que l'on veut.

Partons de l'intégrale (23) et soit comme plus haut $0 \leqq h < \omega_1 + \omega_2 + \cdots + \omega_n$. En intégrant par partie on trouve

$$R^{(n)}_{m+1}(x) = R^{(n)}_m(x) - \frac{B^{(n)}_{m+n}(h)}{(m+n)!} \overset{n}{\underset{\omega_1 \ldots \omega_n}{\triangle}} \varphi^{(m)}(x).$$

Dans cette équation remplaçons m successivement par $m, m-1, m-2, \ldots, 1$ et ajoutons ensemble les équations ainsi obtenues. Il vient

$$R^{(n)}_{m+1}(x) = R^{(n)}_1(x) - \sum_{\nu=1}^{\nu=m} \frac{B^{(n)}_{\nu+n}(h)}{(\nu+n)!} \overset{n}{\underset{\omega_1 \ldots \omega_n}{\triangle}} \varphi^{(\nu)}(x). \tag{24}$$

Considérons l'intégrale

$$R^{(n)}_1(x) = \int_0^\infty \frac{\overline{B}^{(n)}_n(h-t)}{n!} \overset{n}{\underset{\omega_1 \ldots \omega_n}{\triangle}} \varphi'(x+t)\, dt.$$

La fonction sous le signe est discontinue dans les points $t = h + s_1\omega_1 + s_2\omega_2 + \ldots + s_n\omega_n$, où les s_i sont des entiers non négatifs. Elle est continue dans tout autre point situé dans l'intervalle d'intégration. En tenant compte de ce que nous avons dit dans le paragraphe 9 relativement aux discontinuités on trouve, en intégrant par partie

$$\int_0^{\mu+h} \frac{\overline{B}^{(n)}_n(h-t)}{n!} \overset{n}{\underset{\omega_1 \ldots \omega_n}{\triangle}} \varphi'(x+t)\, dt = \int_0^{\mu+h} \frac{B^{(n)}_{n-1}(h-t)}{(n-1)!} \overset{n}{\underset{\omega_1 \ldots \omega_n}{\triangle}} \varphi(x+t)\, dt$$

$$- \frac{B^{(n)}_n(h)}{n!} \overset{n}{\underset{\omega_1 \ldots \omega_n}{\triangle}} \varphi(x) + \frac{\overline{B}^{(n)}_n(-\mu)}{n!} \overset{n}{\underset{\omega_1 \ldots \omega_n}{\triangle}} \varphi(x+h+\mu) + P(\mu)$$

où

$$P(\mu) = (-1)^n \omega_1 \omega_2 \cdots \omega_n \sum \overset{n}{\underset{\omega_1 \ldots \omega_n}{\triangle}} \varphi(x + h + s_1\omega_1 + s_2\omega_2 + \cdots + s_n\omega_n),$$

la sommation étant etendue à toutes les valeurs entières non négatives des s_i qui vérifient l'inégalité

$$s_1\omega_1 + s_2\omega_2 + \cdots + s_n\omega_n < \mu.$$

Faisons tendre μ vers l'infini. Le troisième terme au second membre tend vers zéro. Il vient donc

$$(25)\quad R_1^{(n)}(x) = R_0^{(n)}(x) - \frac{B_n^{(n)}(h)}{n!} \overset{n}{\underset{\omega_1 \dots \omega_n}{\triangle}} \varphi(x) + (-1)^n \omega_1 \omega_2 \cdots \omega_n \sum \overset{n}{\underset{\omega_1 \dots \omega_n}{\triangle}} \varphi(x+h+\Omega)$$

où l'on a posé

$$\Omega = s_1 \omega_1 + s_2 \omega_2 + \cdots + s_n \omega_n .$$

Dans la dernière série la sommation est étendue à toutes les valeurs entières, non négatives des s_i. Cette série est absolument convergente. Il est facile de l'évaluer, On a en effet

$$\sum_{s_1=0}^{s_1=p_1-1} \cdots \sum_{s_n=0}^{s_n=p_n-1} \overset{n}{\underset{\omega_1 \dots \omega_n}{\triangle}} \varphi(x+\Omega) = p_1 p_2 \cdots p_n \overset{n}{\underset{p_1 \omega_1 \dots p_n \omega_n}{\triangle}} \varphi(x),$$

$p_1, p_2, \dots, p_n$ étant des entiers positifs quelconques. Rappelons que la fonction $\varphi(x)$, par hypothèse, tend vers zéro quand x augmente indéfiniment. En faisant tendre les p_i vers l'infini on trouve par conséquent

$$\omega_1 \omega_2 \cdots \omega_n \sum \overset{n}{\underset{\omega_1 \dots \omega_n}{\triangle}} \varphi(x+\Omega) = (-1)^n \varphi(x).$$

L'équation (25) se réduit donc à la suivante

$$R_1^{(n)}(x) = R_0^{(n)}(x) - \frac{B_n^{(n)}(h)}{n!} \overset{n}{\underset{\omega_1 \dots \omega_n}{\triangle}} \varphi(x) + \varphi(x+h).$$

En substituant cette expression dans l'équation (24) on trouvera

$$(26)\quad \varphi(x+h) + \int_0^\infty \frac{B_{n-1}^{(n)}(h-t)}{(n-1)!} \overset{n}{\underset{\omega_1 \dots \omega_n}{\triangle}} \varphi(x+t)\, dt = \sum_{\nu=0}^{\nu=m} \frac{B_{\nu+n}^{(n)}(h)}{(\nu+n)!} \overset{n}{\underset{\omega_1 \dots \omega_n}{\triangle}} \varphi^{(\nu)}(x) + R_{m+1}^{(n)}(x)$$

où

$$R_{m+1}^{(n)}(x) = \int_0^\infty \frac{\bar{B}_{m+n}^{(n)}(h-t)}{(m+n)!} \overset{n}{\underset{\omega_1 \dots \omega_n}{\triangle}} \varphi^{(m+1)}(x+t)\, dt.$$

On vérifie aisément que cette relation est en accord avec la relation (18). En effet, en développant le polynome de Bernoulli suivant les puissances de t on trouvera

$$\frac{B_{n-1}^{(n)}(h-t)}{(n-1)!} = \sum_{\nu=0}^{n-1} \frac{B_\nu^{(n)}(h)}{\nu!} \frac{(-t)^{n-1-\nu}}{(n-1-\nu)!}.$$

Par conséquent on aura

$$R_0^{(n)}(x) = \int_0^\infty \frac{B_{n-1}^{(n)}(h-t)}{(n-1)!} \overset{n}{\underset{\omega_1 \dots \omega_n}{\triangle}} \varphi(x+t)\, dt$$

$$= \sum_{\nu=0}^{n-1} (-1)^{n-1-\nu} \frac{B_\nu^{(n)}(h)}{\nu!} \int_0^\infty \frac{t^{n-1-\nu}}{(n-1-\nu)!} \overset{n}{\underset{\omega_1 \dots \omega_n}{\triangle}} \varphi(x+t)\, dt$$

$$= (-1)^{n-1} \sum_{\nu=0}^{n-1} \frac{B_\nu^{(n)}(h)}{\nu!} \int_0^\infty \cdots \int_0^\infty \overset{n}{\underset{\omega_1 \dots \omega_n}{\triangle}} \varphi^{(\nu)}(x+t_1+\cdots+t_n)\, dt_1 \cdots dt_n$$

$$= \frac{-1}{\omega_1 \omega_2 \cdots \omega_n} \sum_{\nu=0}^{n-1} \frac{B_\nu^{(n)}(h)}{\nu!} \int_0^{\omega_1} dt_1 \int_0^{\omega_2} dt_2 \cdots \int_0^{\omega_n} \varphi^{(\nu)}(x+t_1+\cdots+t_n)\, dt_n.$$

En remplaçant l'intégrale au premier membre de l'équation (26) par cette expression on retombe sur la relation (18).

13. La relation que nous venons de trouver permet de décider de la convergence de certaines séries à n entrées de la forme

$$\sum \varphi(x+\Omega), \tag{27}$$

la sommation étant étendue à toutes les valeurs entières non négatives de $s_1, s_2, \dots, s_n$. Supposons que

(1) la fonction $\varphi(x)$ admet, pour $x \geqq a$, une dérivée continue d'ordre m telle que

$$\lim_{x\to\infty} x^{n+\varepsilon}\varphi^{(m)}(x) = 0, \tag{28}$$

ε étant un nombre positif;

(2) l'intégrale n fois itérée

$$\int_0^\infty dt_n \cdots \int_0^\infty dt_2 \int_0^\infty \varphi(a+t_1+t_2+\cdots+t_n)\,dt_1 \tag{29}$$

est convergente. *Dans ces conditions la série* (27) *sera uniformément convergente* dans l'intervalle* $x \geqq a$. *La série n fois itérée*

$$\sum_{s_n=0}^{\infty}{}' \cdots \sum_{s_2=0}^{\infty}{}' \sum_{s_1=0}^{\infty}{}' \varphi(x+\Omega) \tag{30}$$

convergera aussi et elle aura la même somme que la série (27).

En effet, nous avons vu qu'on a

$$\varphi(x+h) = \sum_{\nu=0}^{\nu=n-1} \frac{B_\nu^{(n)}(h)}{\nu!} \int_0^1 dt_n \cdots \int_0^1 \varphi^{(\nu)}(x+\omega_1 t_1+\cdots+\omega_n t_n)\,dt_1$$
$$+ \sum_{\nu=0}^{\nu=m-1} \frac{B_{\nu+n}^{(n)}(h)}{(\nu+n)!} \overset{n}{\underset{\omega_1\cdots\omega_n}{\triangle}} \varphi^{(\nu)}(x) + R_m^{(n)}(x).$$

On en déduit la relation suivante

$$\sum_{s_n=0}^{p_n-1} \cdots \sum_{s_2=0}^{p_2-1} \sum_{s_1=0}^{p_1-1} \varphi(x+h+\Omega) \tag{31}$$

$$= \frac{1}{\omega_1\omega_2\cdots\omega_n} \sum_{\nu=0}^{\nu=n-1} \frac{B_\nu^{(n)}(h)}{\nu!} \int_0^{p_n\omega_n} dt_n \cdots \int_0^{p_2\omega_2} dt_2 \int_0^{p_1\omega_1} \varphi^{(\nu)}(x+t_1+t_2+\cdots+t_n)\,dt_1$$

$$+ p_1 p_2 \cdots p_n \sum_{\nu=0}^{\nu=m-1} \frac{B_{\nu+n}^{(n)}(h)}{(\nu+n)!} \overset{n}{\underset{p_1\omega_1\cdots p_n\omega_n}{\triangle}} \varphi^{(\nu)}(x) + \sum_{s_n=0}^{p_n-1} \cdots \sum_{s_1=0}^{p_1-1} R_m^{(n)}(x+\Omega),$$

* Ce théorème est une extension d'un théorème, relativement aux séries à entrée simple, donné par Bromwich (Proceedings of the London Mathematical Society, ser. 2, vol. 6 (1908), pp. 327-338) et Hardy (idem ser. 2, vol. 9 (1911), pp. 126-144). Voir aussi: Acta Mathematica, vol. 44, p. 103.

$p_1, p_2, \ldots, p_n$ étant des entiers positifs quelconques. Il s'agit de démontrer que le second membre tend vers une limite quand les entiers $p_1, \ldots, p_n$ tendent vers l'infini simultanément et indépendamment l'un de l'autre.

Considérons d'abord le terme complémentaire. On a

$$R_m^{(n)}(x) = \int_0^\infty \frac{\bar{B}_{m+n-1}^{(n)}(h-t)}{(m+n-1)!} \overset{n}{\underset{\omega_1 \ldots \omega_n}{\triangle}} \varphi^{(m)}(x+t)\,dt$$

et par conséquent

$$(32) \quad \sum_{s_n=0}^{p_n-1} \cdots \sum_{s_2=0}^{p_2-1} \sum_{s_1=0}^{p_1-1} R_m^{(n)}(x+\Omega)$$

$$= p_1 p_2 \cdots p_n \int_0^\infty \frac{\bar{B}_{m+n-1}^{(n)}(h-t)}{(m+n-1)!} \overset{n}{\underset{p_1\omega_1 \ldots p_n\omega_n}{\triangle}} \varphi^{(m)}(x+t)\,dt.$$

Cette intégrale peut se décomposer en une somme de 2^n intégrales dont une est la suivante

$$(33) \quad \frac{(-1)^n}{\omega_1 \omega_2 \cdots \omega_n} \int_0^\infty \frac{\bar{B}_{m+n-1}^{(n)}(h-t)}{(m+n-1)!} \varphi^{(m)}(x+t)\,dt$$

pendant que les autres sont de la forme

$$(34) \quad P = \frac{\pm 1}{\omega_1 \omega_2 \cdots \omega_n} \int_0^\infty \frac{\bar{B}_{m+n-1}^{(n)}(h-t)}{(m+n-1)!} \varphi^{(m)}(x+p+t)\,dt$$

p étant un nombre qui tend vers l'infini quand tous les nombres $p_1, p_2, \cdots, p_n$ tendent vers l'infini. L'intégrale (33) ne dépend pas des p_i. Considérons les intégrales (34). On a

$$\left| \bar{B}_{m+n-1}^{(n)}(h-t) \right| < C t^{m-1},$$

C étant une constante. En tenant compte de l'équation (28) on sait trouver une constante C_1 telle que

$$|P| < C_1 \int_0^\infty \frac{t^{n-1}\,dt}{(x+p+t)^{n+\varepsilon}} < C_1 \int_0^\infty \frac{dt}{(x+p+t)^{1+\varepsilon}} = \frac{C_1}{\varepsilon} \frac{1}{(x+p)^\varepsilon}.$$

Il en résulte que les intégrales (34) tendent vers zéro quand tous les nombres p_i tendent vers l'infini et cela uniformément pour $x \geqq a$. Le dernier terme au second membre de l'équation (31) tend donc uniformément vers l'intégrale (33).

Posons

$$(35) \qquad f(x) = (-1)^n \int_0^\infty dt_n \cdots \int_0^\infty dt_2 \int_0^\infty \varphi(x + t_1 + t_2 + \cdots + t_n)\, dt_1.$$

On a évidemment

$$f(x) = (-1)^n \int_x^\infty dt_n \int_{t_n}^\infty dt_2 \int_{t_2}^\infty \varphi(t_1)\, dt_1.$$

De l'hypothèse (2) il résulte que cette intégrale convergera pour $x \geqq a$ et que

$$\lim_{x \to \infty} f(x) = 0.$$

En dérivant n fois par rapport à x on trouve, pour $x > a$

$$(36) \qquad f^{(n)}(x) = \varphi(x).$$

De l'hypothèse (1) il résulte par conséquent que

$$\lim_{x \to \infty} f^{(n+m)}(x) = 0.$$

On a donc, en vertu du théorème de Littlewood et Hardy, mentionné plus haut,

$$(37) \qquad \lim_{x \to \infty} f^{(\nu)}(x) = 0, \qquad (\nu = 0, 1, 2, \ldots, m+n).$$

Cela posé considérons l'intégrale

$$\frac{1}{\omega_1 \omega_2 \cdots \omega_n} \int_0^{p_n \omega_n} \cdots \int_0^{p_2 \omega_2} \int_0^{p_1 \omega_1} \varphi^{(\nu)}(x + t_1 + t_2 + \cdots + t_n)\, dt_1\, dt_2 \cdots dt_n,$$

$$0 \leqq \nu < n.$$

De l'équation (36) il résulte que cette intégrale est égale à

$$p_1 p_2 \cdots p_n \underset{p_1 \omega_1 \cdots p_n \omega_n}{\overset{n}{\triangle}} f^{(\nu)}(x).$$

Faisons tendre les entiers p_i vers l'infini. Puisque $\lim_{x \to \infty} f^{(\nu)}(x) = 0$ l'intégrale tendra uniformement vers une limite* et il vient

$$f^{(\nu)}(x) = (-1)^n \int_0^\infty \cdots \int_0^\infty \int_0^\infty \varphi^{(\nu)}(x + t_1 + t_2 + \cdots + t_n)\, dt_1\, dt_2 \cdots dt_n.$$

Si $\nu > 0$ on peut réduire le degré de multiplicité de cette intégrale et l'on trouvera

$$\int_0^\infty \cdots \int_0^\infty \varphi^{(\nu)}(x + t_1 + t_2 + \cdots + t_n)\, dt_1\, dt_2 \cdots dt_n$$

$$= (-1)^\nu \int_0^\infty \cdots \int_0^\infty \varphi\,(x + t_{\nu+1} + t_{\nu+2} + \cdots + t_n)\, dt_{\nu+1} \cdots dt_n.$$

Le premier terme au second membre de l'équation (31) tend donc uniformément vers une limite. Le second terme ne fait pas de difficulté. On voit immédiatement que ce terme tend uniformément vers

* La convergence de l'intégrale itérée (35) entraine donc la convergence de l'intégrale multiple

$$\int_0^\infty \cdots \int_0^\infty \int_0^\infty \varphi\,(x + t_1 + t_2 + \cdots + t_n)\, dt_1\, dt_2 \cdots dt_n,$$

mais l'inverse n'a pas lieu. La dernière intégrale sera convergente par exemple si

$$\varphi\,(x) = x^{n-1} \sin(x^2),$$

mais l'intégrale

$$\int_1^\infty \varphi\,(x)\, dx$$

divergera si $n \geqq 2$, et l'intégrale (35) sera donc *a fortiori* divergente.

$$\frac{(-1)^n}{\omega_1\,\omega_2\cdots\omega_n}\sum_{\nu=0}^{\nu=m-1}\frac{B_{\nu+n}^{(n)}(h)}{(\nu+n)!}\,\varphi^{(\nu)}(x)$$

parce que

$$\lim_{x\to\infty}\varphi^{(\nu)}(x)=0, \qquad (\nu=0,1,2,\ldots,m-1).$$

La convergence de la série (27) est ainsi démontrée et l'on voit qu'elle se représente par l'expression suivante

$$(-1)^n\,\omega_1\,\omega_2\cdots\omega_n\sum\varphi(x+h+\Omega)$$

$$=\sum_{\nu=0}^{\nu=n-1}\frac{B_\nu^{(n)}(h)}{\nu!}(-1)^{n-\nu}\int_0^\infty\cdots\int_0^\infty\varphi(x+t_1+t_2+\cdots+t_{n-\nu})\,dt_1\,dt_2\cdots dt_{n-\nu}$$

$$+\sum_{\nu=0}^{\nu=m-1}\frac{B_{\nu+n}^{(n)}(h)}{(\nu+n)!}\,\varphi^{(\nu)}(x)+\int_0^\infty\frac{\bar{B}_{m+n-1}^{(n)}(h-t)}{(m+n-1)!}\,\varphi^{(m)}(x+t)\,dt.$$

Cette équation peut aussi s'écrire sous la forme:

$$(-1)^{(n)}\,\omega_1\,\omega_2\cdots\omega_n\sum\varphi(x+h+\Omega)$$

$$=\sum_{\nu=0}^{\nu=n+m-1}\frac{B_\nu^{(n)}(h)}{\nu!}f^{(\nu)}(x)+\int_0^\infty\frac{\bar{B}_{m+n-1}^{(n)}(h-t)}{(m+n-1)}\,f^{(m+n)}(x+t)\,dt.$$

Remarquons enfin que tout ce que nous venons de dire reste vraie si, dans l'équation (31), on fait tendre d'abord p_1, puis p_2, ... et enfin p_n vers l'infini. Le second membre de cette équation tendra vers une limite et vers la même limite que dans le cas où ces nombres tendent simultanément vers l'infini. La série n fois itérée (30) sera donc convergente.

Soit en particulier

$$\varphi(x)=x^{-n-\beta}$$

et supposons que x n'est pas négatif ou nul. On voit que la série à n entrées

$$\sum\frac{1}{(x+\Omega)^{n+\beta}}$$

et la série itérée

$$\sum_{s_n=0}^{\infty}\sum_{s_2=0}^{\infty}\sum_{s_1=0}^{\infty}\frac{1}{(x+\Omega)^{n+\beta}}$$

seront toutes les deux convergentes ou divergentes suivant que la partie réelle de β est positive ou non. Voici un autre exemple qui est assez intéressant. Soit

$$\varphi(x)=\frac{\sin(x^{\alpha})}{x^{\beta}},\qquad 0<\alpha<1.$$

En dérivant m fois par rapport à x on voit que

$$|x^{\beta+m(1-\alpha)}\varphi^{(m)}(x)|$$

reste plus petit qu'une constante quand x augmente indéfiniment. L'équation (28) est donc satisfaite si

$$m>\Re\left(\frac{n-\beta}{1-\alpha}\right).$$

Considérons l'intégrale

$$\int_x^{\infty}\frac{\sin t^{\alpha}}{t^{\beta}}\,dt.$$

Cette intégrale convergera si $\Re(\beta)>1-\alpha$, et elle divergera si $\Re(\beta)\leqq 1-\alpha$. En intégrant par partie on trouve

$$\int_x^{\infty}\frac{\sin t^{\alpha}}{t^{\beta}}\,dt=\frac{1}{\alpha}\,\frac{\cos x^{\alpha}}{x^{\alpha+\beta-1}}+\frac{1}{\alpha}\left(1+\frac{\beta-1}{\alpha}\right)\frac{\sin x^{\alpha}}{x^{2\alpha+\beta-1}}-\cdots$$

$$+\frac{(-1)^{p-1}}{\alpha}\left(1+\frac{\beta-1}{\alpha}\right)\left(2+\frac{\beta-1}{\alpha}\right)\cdots\left(2p-1+\frac{\beta-1}{\alpha}\right)\frac{\sin x^{\alpha}}{x^{2p\alpha+\beta-1}}$$

$$+(-1)^{p}\left(1+\frac{\beta-1}{\alpha}\right)\left(2+\frac{\beta-1}{\alpha}\right)\cdots\left(2p+\frac{\beta-1}{\alpha}\right)\int_x^{\infty}\frac{\sin t^{\alpha}\,dt}{t^{2p\alpha+\beta}}.$$

En substituant cette expression dans l'intégrale

$$\int_x^\infty dt_2 \int_{t_2}^\infty \frac{\sin t_1^\alpha}{t_1^\beta}\, dt,$$

on voit qu'elle convergera si $\Re(\beta) > 2(1-\alpha)$. Et en choisissant p suffisamment grand on démontre aisément que la condition nécessaire et suffisante pour que l'intégrale n fois itérée soit convergente c'est que $\Re(\beta) > n(1-\alpha)$. De notre théorème il résulte donc que l'intégrale*

$$(38) \qquad \int_0^\infty \cdots \int_0^\infty \int_0^\infty \frac{\sin(x+t_1+t_2+\cdots+t_n)^\alpha}{(x+t_1+t_2+\cdots+t_n)^\beta}\, dt_1\, dt_2 \cdots dt_n$$

et que les séries

$$\sum' \frac{\sin(x+\Omega)^\alpha}{(x+\Omega)^\beta},$$

$$\sum_{s_n=0}^{\infty}{}' \cdots \sum_{s_2=0}^{\infty}{}' \sum_{s_1=0}^{\infty}{}' \frac{\sin(x+\Omega)^\alpha}{(x+\Omega)^\beta}$$

toutes les trois convergeront, si $\Re(\beta) > n(1-\alpha)$, et divergeront si $\Re(\beta) \leqq n(1-\alpha)$.

14. Si l'intégrale multiple est absolument convergente on peut l'évaluer d'une manière bien simple. En effet, considérons l'intégrale

$$P_n = \int\int \cdots \int \varphi(x+t_1+t_2+\cdots+t_n)\, dt_1\, dt_2 \cdots dt_n$$

étendue au continuum défini par les inégalites

$$t_1 \geqq 0, \quad t_2 \geqq 0, \ldots, t_n \geqq 0,$$

$$t_1 + t_2 + \cdots + t_n \leqq h.$$

* Nous avons supposé que $1 > \alpha > 0$. Si $\alpha \geqq 1$ on voit aisément que la condition pour que l'intégrale (38) converge est encore celle qui a été donnée dans le texte. Mais cela n'est plus vrai pour les deux séries.

On a évidemment

$$P_n = \int_0^h \frac{t^{n-1}}{(n-1)!}\,\varphi(x+t)\,dt.$$

En effet cette équation est vraie, si $n = 1$. Supposons qu'elle soit vraie pour P_{n-1}. On aura alors

$$P_n = \int_0^h dt_n \int_{t_n}^h \frac{(t-t_n)^{n-2}}{(n-2)!}\,\varphi(x+t)\,dt$$

$$= \int_0^h \varphi(x+t)\,dt \int_0^t \frac{(t-t_n)^{n-2}}{(n-2)!}\,dt_n = \int_0^h \frac{t^{n-1}}{(n-1)!}\,\varphi(x+t)\,dt.$$

Faisons tendre h vers l'infini dans l'équation (39), il vient

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \varphi(x+t_1+t_2+\cdots+t_n)\,dt_1\,dt_2\cdots dt_n$$

$$= \int_0^\infty \frac{t^{n-1}}{(n-1)!}\,\varphi(x+t)\,dt,$$

pourvu que l'intégrale au premier membre soit absolument convergente. Dans ces conditions on a par conséquent aussi

$$\int_x^\infty dt_n \cdots \int_{t_3}^\infty dt_2 \int_{t_2}^\infty \varphi(t_1)\,dt_1 = \int_0^\infty \frac{t^{n-1}}{(n-1)!}\,\varphi(x+t)\,dt,$$

ce qu'on trouve encore en renversant l'ordre des intégrations. Si $\varphi(x)$ est positive à partir d'une certaine valeur de x, l'intégrale au premier membre sera donc convergente ou divergente suivant que l'intégrale au second membre aura ou non un sens. On a par conséquent le théorème suivant:

Admettons que la fonction $\varphi(x)$ est positive, pour $x \geq a$, et qu'elle admet une dérivée continue d'ordre m telle que

$$\lim\, x^{n+\varepsilon}\,\varphi^{(m)}(x) = 0, \qquad \varepsilon > 0. \tag{40}$$

Alors, pour que la série à n entrées

$$\sum \varphi(x+\Omega) \tag{41}$$

converge il faut et il suffit que l'intégrale

$$\int_a^\infty x^{n-1}\varphi(x)\,dx \tag{42}$$

converge. Supposons par exemple que

$$\varphi(x) = \frac{1}{x^n \log x \log_2 x \cdots \log_r x\,(\log_r x)^\varepsilon}.$$

On voit que la série (41) converge ou diverge suivant que ε est positif ou négatif. Supposons en second lieu que

$$\varphi(x) = \frac{1}{x^\beta\left(1+x^\alpha(\sin \pi\sqrt{x})^2\right)}, \qquad \text{où } \tfrac{1}{2} > \alpha > 0, \qquad \beta > 0.$$

Pour les valeurs positives et très grandes de x cette fonction est en général du même ordre de grandeur que $x^{-\beta-\alpha}$, mais, si x est le carré d'un entier, $\varphi(x)$ est égale à $x^{-\beta}$. On peut trouver un entier m tel que l'égalité (40) soit satisfaite, et l'on peut aisément décider de la convergence de l'intégrale (42). On a en effet

$$\frac{2}{(p+1)^{2\beta+1}\sqrt{1+(p+1)^{2\alpha}}} < \int_{p^2}^{(p+1)^2} \varphi(x)\frac{dx}{x} < \frac{2}{p^{2\beta+1}\sqrt{1+p^{2\alpha}}},$$

p étant un entier positif quelconque. L'intégrale (42) est par conséquent convergente, si $\beta > n-\frac{\alpha}{2}$, et divergente, si $\beta \leqq n-\frac{\alpha}{2}$. Il en est donc de même de la série (41).

Si la fonction $\varphi(x)$ est positive à partir d'une certaine valeur de x, continuellement décroissante et tendant vers zéro quand x augmente indéfiniment, on n'a pas besoin de faire des hypothèses relativement aux dérivées de $\varphi(x)$

car on peut comparer directement la série et l'intégrale et l'on voit immédiatement que *la série à n entrées*

$$\sum \varphi(x+\Omega)$$

sera convergente ou divergente, suivant que l'intégrale

$$\int_a^\infty x^{n-1}\varphi(x)\,dx$$

aura ou non un sens.

En ce cas particulier le théorème se déduit d'un théorème classique de Maclaurin* et de Cauchy† qui a été étendu à des séries multiples par plusieurs auteurs‡.

15. Considérons la fonction génératrice des polynomes de Bernoulli§

$$\frac{\omega_1\,\omega_2\cdots\omega_n\,\eta^n e^{-\eta h}}{(1-e^{-\omega_1\eta})(1-e^{-\omega_2\eta})\cdots(1-e^{-\omega_n\eta})} = \sum_{\nu=0}^{\infty}\frac{(-\eta)^\nu}{\nu!}B_\nu^{(n)}(h).$$

Si l'on arrête la série au second membre à un certain terme, on peut trouver une expression simple du reste. En effet, reprenons la formule

$$f^{(n)}(x+h) = \sum_{\nu=0}^{\nu=m}\frac{B_\nu^{(n)}(h)}{\nu!}\underset{\omega_1\cdots\omega_n}{\overset{n}{\triangle}} f^{(\nu)}(x) + \int_0^\infty \frac{\bar{B}_m^{(n)}(h-t)}{m!}\underset{\omega_1\cdots\omega_n}{\overset{n}{\triangle}} f^{(m+1)}(x+t)\,dt,$$

où m est un entier supérieur ou égal à n. Posons $f(x) = e^{-\eta x}$, η étant un nombre positif. On aura

$$\underset{\omega_1\cdots\omega_n}{\overset{n}{\triangle}} f^{(\nu)}(x) = (-\eta)^\nu e^{-\eta x}\frac{(e^{-\eta\omega_1}-1)(e^{-\eta\omega_2}-1)\cdots(e^{-\eta\omega_n}-1)}{\omega_1\,\omega_2\cdots\omega_n}.$$

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* *A Treatise of Fluxions*, vol. I, Edinburgh, 1742, pp. 289–290.

† *Sur la convergence des séries*, *Œuvres*, sér. 2, vol. 7, Paris, 1889, pp. 267–279.

‡ É. Picard, *Traité d'Analyse*, vol. I, 2e édition, Paris, 1901, pp. 288–304. É. Goursat, *Cours d'Analyse mathématique*, vol. I, 2e édition, Paris, 1910, pp. 424–426. T. Bromwich, *An Introduction to the Theory of Infinite Series*, London 1908, p. 80; *Report British Association*, 1906, pp. 493–494.

§ Acta Mathematica, vol. 43, p. 183.

En substituant cette expression dans l'équation précédente on trouvera

$$\frac{\omega_1\omega_2\cdots\omega_n\eta^n e^{-\eta h}}{(1-e^{-\omega_1\eta})(1-e^{-\omega_2\eta})\cdots(1-e^{-\omega_n\eta})}$$

$$=\sum_{\nu=0}^{\nu=m}\frac{B_\nu^{(n)}(h)}{\nu!}(-\eta)^\nu+(-\eta)^{m+1}\int_0^\infty\frac{\bar{B}_m^{(n)}(h-t)}{m!}e^{-\eta t}dt.$$

De cette relation on peut déduire un lemme qui nous sera utile plus loin. En développant le premier membre suivant les puissances de η on trouve

$$\int_0^\infty\frac{\bar{B}_m^{(n)}(h-t)}{m!}e^{-\eta t}dt=\sum_{\nu=0}^\infty\frac{B_{\nu+m+1}^{(n)}(h)}{(\nu+m+1)!}(-\eta)^\nu.$$

Dérivons p fois par rapport à η, il vient

$$\int_0^\infty\frac{\bar{B}_m^{(n)}(h-t)}{m!}t^p e^{-\eta t}dt=\sum_{\nu=0}^\infty\frac{\bar{B}_{\nu+m+p+1}^{(n)}(h)}{(\nu+m+p+1)!}\frac{(\nu+p)!}{\nu!}(-\eta)^\nu.$$

L'intégrale au premier membre est divergente si $\eta=0$. Mais elle représente, pour $\eta>0$, une fonction analytique de η qui est holomorphe dans le point $\eta=0$. L'intégrale tend donc vers une limite, quand η tend vers zéro, et l'on trouve

$$\lim_{\eta\to 0}\int_0^\infty\bar{B}_m^{(n)}(h-t)t^p e^{-\eta t}dt=\frac{m!\,p!}{(m+p+1)!}B_{m+p+1}^{(n)}(h).$$

Nous avons supposé $m\geqq n$. Si $m<n$ l'intégrale tend vers l'infini quand η tend vers zéro.

La solution principale $F_n(x)$

16. Admettons que la fonction $\varphi(x)$ satisfait à la condition du paragraphe 8. Considérons l'expression suivante

$$(43)\qquad\int_a^\infty\frac{B_{n-1}^{(n)}(x-z)}{(n-1)!}\varphi(z)e^{-\eta z}dz+(-1)^n\omega_1\omega_2\cdots\omega_n\sum\varphi(x+\Omega)e^{-\eta(x+\Omega)},$$

la sommation étant étendue à toutes les valeurs entières non négatives de $s_1, s_2, \ldots, s_n$. La série et l'intégrale convergent évidemment pour toute valeur positive de η. Je dis que l'expression (43) tend vers une limite quand le nombre positif η tend vers zéro, et cela uniformément quand x reste dans un intervalle fini quelconque $a \leqq x \leqq b$. Cette limite représente donc une fonction continue de x que je désignerais par $F_n(x \mid \omega_1, \omega_2, \ldots, \omega_n)$. En effet, soient $p_1, p_2, \ldots, p_n$ des entiers positifs quelconques et posons

$$f(x) = \varphi(x) e^{-\eta x}.$$

Remplaçons dans l'équation (26) $\varphi(x)$ par $f(x)$ et x par $x+\Omega$; donnons aux s_i les valeurs $0, 1, 2, \ldots, p_i - 1$ et ajoutons ensemble les équations ainsi obtenues. On trouvera

$$\sum_{s_n=0}^{p_n-1} \cdots \sum_{s_2=0}^{p_2-1} \sum_{s_1=0}^{p_1-1} f(x+h+\Omega) + p_1 p_2 \cdots p_n \int_0^\infty \frac{B_{n-1}^{(n)}(h-t)}{(n-1)!} \overset{n}{\underset{p_1\omega_1 \cdots p_n\omega_n}{\triangle}} f(x+t)\, dt$$

$$= p_1 p_2 \cdots p_n \sum_{\nu=0}^{\nu=m} \frac{B_{\nu+n}^{(n)}(h)}{(\nu+n)!} \overset{n}{\underset{p_1\omega_1 \cdots p_n\omega_n}{\triangle}} f^{(\nu)}(x)$$

$$+ p_1 p_2 \cdots p_n \int_0^\infty \frac{\overline{B}_{m+n}^{(n)}(h-t)}{(m+n)!} \overset{n}{\underset{p_1\omega_1 \cdots p_n\omega_n}{\triangle}} f^{(m+1)}(x+t)\, dt.$$

Faisons tendre les entiers $p_1, p_2, \ldots, p_n$ vers l'infini. Il vient

$$\int_a^\infty \frac{B_{n-1}^{(n)}(x+h-z)}{(n-1)!} \varphi(z) e^{-\eta z}\, dz + (-1)^n \omega_1 \omega_2 \cdots \omega_n \sum \varphi(x+h+\Omega) e^{-\eta(x+h+\Omega)}$$

$$= \int_a^x \frac{B_{n-1}^{(n)}(x+h-z)}{(n-1)!} \varphi(z) e^{-\eta z}\, dz + \sum_{\nu=0}^{\nu=m} \frac{B_{\nu+n}^{(n)}(h)}{(\nu+n)!} D_x^\nu [\varphi(x) e^{-\eta x}]$$

$$+ \int_0^\infty \frac{\overline{B}_{m+n}^{(n)}(h-t)}{(m+n)!} D_x^{m+1} [\varphi(x+t) e^{-\eta(x+t)}]\, dt.$$

Faisons ensuite tendre η vers zéro. En tenant compte du lemme du paragraphe 15 et en raisonnant tout à fait de la même manière que dans le paragraphe 8 on démontre que le second membre tendra uniformément vers une limite et que cette limite est égale à l'expression qu'on obtient quand on pose η égal à zéro. On aura par conséquent

$$(44)\quad F_n(x+h\,|\,\omega_1,\omega_2,\ldots,\omega_n) = \int_a^x \frac{B_{n-1}^{(n)}(x+h-z)}{(n-1)!}\,\varphi(z)\,dz$$

$$+\sum_{\nu=0}^{\nu=m}\frac{B_{\nu+n}^{(n)}(h)}{(\nu+n)!}\,\varphi^{(\nu)}(x)+\int_0^\infty \frac{\bar{B}_{m+n}^{(n)}(h-t)}{(m+n)!}\,\varphi^{(m+1)}(x+t)\,dt.$$

En remarquant que le polynome de Bernoulli se développe de la manière suivante

$$\frac{B_{n-1}^{(n)}(x+h-z)}{(n-1)!} = \sum_{\nu=0}^{n-1}\frac{B_\nu^{(n)}(h)}{\nu!}\,\frac{(x-z)^{n-1-\nu}}{(n-1-\nu)!},$$

on voit que la première intégrale au second membre de l'équation (44) est égale à

$$\int_a^x \frac{B_{n-1}^{(n)}(x+h-z)}{(n-1)!}\,\varphi(z)\,dz = \sum_{\nu=0}^{n-1}\frac{B_\nu^{(n)}(h)}{\nu!}\int_a^x \frac{(x-z)^{n-1-\nu}}{(n-1-\nu)!}\,\varphi(z)\,dz.$$

Si l'on pose

$$f(x) = \int_a^x \frac{(x-z)^{n-1}}{(n-1)!}\,\varphi(z)\,dz,$$

l'équation (44) peut donc s'écrire comme il suit

$$F_n(x+h\,|\,\omega_1,\omega_2,\ldots,\omega_n)$$

$$=\sum_{\nu=0}^{n+m}\frac{B_\nu^{(n)}(h)}{\nu!}\,f^{(\nu)}(x)+\int_0^\infty \frac{\bar{B}_{m+n}^{(n)}(h-t)}{(m+n)!}\,f^{(n+m+1)}(x+t)\,dt.$$

On vérifie aisément que la fonction $F_n(x)$ est une solution de l'équation (1) paragraphe 1. En effet, en appliquant l'opération $\underset{\omega_n}{\triangle}$ à l'expression (43) et en faisant tendre η vers zéro on trouvera

$$\underset{\omega_n}{\triangle} F_n(x \mid \omega_1, \omega_2, \ldots, \omega_n) = \lim_{\eta \to 0} \left[\int_a^\infty \frac{B_{n-2}^{(n-1)}(x-z)}{(n-2)!} \varphi(z)\, e^{-\eta z}\, dz \right.$$

$$\left. + (-1)^{n-1} \omega_1 \ldots \omega_{n-1} \sum \varphi(x + s_1\omega_1 + \cdots + s_{n-1}\omega_{n-1})\, e^{-\eta(x+s_1\omega_1+\cdots+s_{n-1}\omega_{n-1})} \right].$$

C'est-à-dire que

$$\underset{\omega_n}{\triangle} F_n(x \mid \omega_1, \omega_2, \ldots, \omega_n) = F_{n-1}(x \mid \omega_1, \omega_2, \ldots, \omega_{n-1}).$$

On en conclut que

$$\overset{2}{\underset{\omega_{n-1}\,\omega_n}{\triangle}} F_n(x \mid \omega_1, \ldots, \omega_n) = F_{n-2}(x \mid \omega_1, \ldots, \omega_{n-2}),$$

. . . . . .

$$\overset{n-1}{\underset{\omega_2 \ldots \omega_n}{\triangle}} F_n(x \mid \omega_1, \ldots, \omega_n) = F_1(x \mid \omega_1),$$

et puisque

$$\underset{\omega_1}{\triangle} F_1(x \mid \omega_1) = \varphi(x)$$

on voit que $F_n(x \mid \omega_1, \ldots, \omega_n)$ est une solution de l'équation

$$\overset{n}{\underset{\omega_1 \ldots \omega_n}{\triangle}} F_n(x) = \varphi(x).$$

Je désigne cette solution comme *la solution principale.*

III.

Valeurs asymptotiques des solutions principales
Développements en séries

17. Je veux maintenant faire voir comment se comportent nos solutions $G_n(x)$ et $F_n(x)$ pour les valeurs positives et très grandes de x. Par hypothèse il existe un entier m tel que

(1) $$\lim_{x \to \infty} x^{m+\varepsilon} \varphi^{(m+1)}(x) = 0$$

pour toute valeur positive de ε. Soit r le plus petit entier positif tel que

$$\lim_{x\to\infty} \varphi^{(r)}(x) = 0. \tag{2}$$

Posons pour abréger

$$P(x) = \sum_{\nu=0}^{r-1} \frac{E_\nu^{(n)}(h)}{\nu!}\varphi^{(\nu)}(x-h), \tag{3}$$

$$Q(x) = \sum_{\nu=0}^{n+r-1} \frac{B_\nu^{(n)}(h)}{\nu!} f^{(\nu)}(x-h), \tag{4}$$

où

$$f(x) = \int_a^x \frac{(x-z)^{n-1}}{(n-1)!}\varphi(z)\,dz.$$

En désignant par h un nombre quelconque dans l'intervalle $0 \leqq h < \omega_1+\omega_2+\cdots+\omega_n$ nous avons démontré que

$$G_n(x\,|\,\omega_1, \omega_2, \ldots, \omega_n) \tag{5}$$

$$= P(x) + \sum_{\nu=r}^{\nu=m} \frac{E_\nu^{(n)}(h)}{\nu!}\varphi^{(\nu)}(x-h) + \int_{-h}^{\infty} \frac{\bar{E}_m^{(n)}(-t)}{m!}\varphi^{(m+1)}(x+t)\,dt,$$

$$F_n(x\,|\,\omega_1, \omega_2, \ldots, \omega_n) \tag{6}$$

$$= Q(x) + \sum_{\nu=r}^{\nu=m} \frac{B_{\nu+n}^{(n)}(h)}{(\nu+n)!}\varphi^{(\nu)}(x-h) + \int_{-h}^{\infty} \frac{\bar{B}_{m+n}^{(n)}(-t)}{(m+n)!}\varphi^{(m+1)}(x+t)\,dt.$$

Il résulte des équations (1) et (2) que le second terme au second membre tend vers zéro quand x augmente indéfiniment. En tenant compte de ce que nous avons dit dans le paragraphe 2 relativement à la fonction $\bar{E}$ on voit qu'on sait trouver une constante C telle que

$$\left| \int_{-h}^{\infty} \bar{E}_m^{(n)}(-t)\varphi^{(m+1)}(x+t)\,dt \right| < C \int_0^{\infty} t^{n-1}\,|\,\varphi^{(m+1)}(x-h+t)\,|\,dt.$$

Mais on a en vertu de l'équation (1)

$$\int_0^\infty t^{n-1}|\varphi^{(m+1)}(x-h+t)|dt<C_1\int_0^\infty \frac{t^{n-1}dt}{(x-h+t)^{n+\varepsilon}}=\frac{C_1}{(x-h)^\varepsilon}\int_0^\infty \frac{t^{n-1}dt}{(1+t)^{n+\varepsilon}}.$$

L'intégrale au second membre de l'équation (5) tend donc vers zéro quand x augmente indéfiniment. On a par conséquent

$$\lim_{x\to\infty}[G_n(x\,|\,\omega_1,\omega_2,\ldots,\omega_n)-P(x)]=0. \tag{7}$$

De l'équation (6) on conclut de même que

$$\lim_{x\to\infty}[F_n(x\,|\,\omega_1,\omega_2,\ldots,\omega_n)-Q(x)]=0. \tag{8}$$

Si l'on fait en particulier $h=0$ les expressions de P et de Q se réduisent aux suivantes

$$P(x)=\sum_{\nu=0}^{r-1}\frac{C_\nu^{(n)}}{2^\nu\,\nu!}\varphi^{(\nu)}(x) \tag{9}$$

et

$$Q(x)=\sum_{\nu=0}^{n+r-1}\frac{B_\nu^{(n)}}{\nu!}f^{(\nu)}(x). \tag{10}$$

Mais si l'on pose $h=\dfrac{\omega_1+\omega_2+\cdots+\omega_n}{2}$ on aura

$$P(x)=\sum_{\nu=0}^{\leqq\frac{r-1}{2}}\frac{E_{2\nu}^{(n)}}{2^{2\nu}(2\nu)!}\varphi^{(2\nu)}\left(x-\frac{\omega_1+\omega_2+\cdots+\omega_n}{2}\right), \tag{11}$$

$$Q(x)=\sum_{\nu=0}^{\leqq\frac{n+r-1}{2}}\frac{D_{2\nu}^{(n)}}{2^{2\nu}(2\nu)!}f^{(2\nu)}\left(x-\frac{\omega_1+\omega_2+\cdots+\omega_n}{2}\right). \tag{12}$$

Soit par exemple $\varphi(x)=\log x$. On aura $r=1$ et l'équation (7) se réduira à la suivante

$$\lim_{x\to\infty}[G_n(x\,|\,\omega_1,\omega_2,\ldots,\omega_n)-\log x]=0.$$

18. On peut souvent obtenir la valeur asymptotique avec une approximation plus grande. Soit p un nombre positif et admettons pour un moment qu'on peut choisir m tel que

$$\lim_{x\to\infty} x^{n+p+\epsilon}\varphi^{(m+1)}(x) = 0.$$

Soit maintenant r le plus petit entier positif tel que

$$\lim_{x\to\infty} x^{p}\varphi^{(r)}(x) = 0.$$

Dans ces conditions il résulte de notre analyse que

$$\lim_{x\to\infty} x^{p}[G_n(x) - P(x)] = 0,$$

$$\lim_{x\to\infty} x^{p}[F_n(x) - Q(x)] = 0.$$

Il arrive donc que ces équations sont satisfaites quelque grand que soit p pourvu qu'on ait choisi r suffisamment grand.

19. Revenons aux hypothèses du paragraphe 17. Je vais indiquer, pour la fonction G_n, une nouvelle expression limite qui est assez remarquable. De l'équation

$$\overset{n}{\underset{\omega_1\dots\omega_n}{\nabla}} G_n(x) = \varphi(x)$$

on tire immédiatement la relation identique suivante

$$0 = \sum_{s_n=0}^{p_n-1}\cdots\sum_{s_2=0}^{p_2-1}\sum_{s_1=0}^{p_1-1}(-1)^{s_1+s_2+\dots+s_n}\varphi(x+\Omega) - \overset{n}{\underset{p_1\omega_1\dots p_n\omega_n}{\nabla}} G_n(x),$$

$p_1, p_2, \dots, p_n$ étant des entiers positifs impairs. En ajoutant et en soustrayant l'expression $P(x) - 2^n \overset{n}{\nabla} P(x)$ dans le second membre, cette relation peut s'écrire comme il suit*

* Quand il n'y a pas lieu à équivoque j'écris pour abréger $\overset{n}{\underset{\omega_i}{\nabla}}$ au lieu de $\overset{n}{\underset{\omega_1\dots\omega_n}{\nabla}}$.

$$G_n(x) = 2^n \sum_{s_n=0}^{p_n-1} \cdots \sum_{s_2=0}^{p_2-1} \sum_{s_1=0}^{p_1-1} (-1)^{s_1+s_2+\cdots+s_n} \varphi(x+\Omega) + P(x)$$

$$-2^n \overset{n}{\underset{p_i\omega_i}{\nabla}} P(x) + \left[G_n(x) - P(x) - 2^n \overset{n}{\underset{p_i\omega_i}{\nabla}} [G_n(x) - P(x)] \right].$$

Faisons tendre tous le nombres p_i vers l'infini. Le dernier terme au second membre tend uniformément vers zéro en vertu de l'équation (7). On a par conséquent

$$(13)\quad G_n(x) = \lim_{p_1,\ldots,p_n\to\infty} \left[2^n \sum_{s_n=0}^{p_n-1} \cdots \sum_{s_2=0}^{p_2-1} \sum_{s_1=0}^{p_1-1} (-1)^{s_1+s_2+\cdots+s_n} \varphi(x+\Omega) \right.$$

$$\left. + P(x) - 2^n \overset{n}{\underset{p_i\omega_i}{\nabla}} P(x) \right].$$

C'est le résultat que j'avais en vue. Si l'on suppose que les dérivées de $\varphi(x)$ qui entrent en considération sont continues pour $x \geqq b$, on voit que cette égalité a lieu uniformément dans l'intervalle $x \geqq b+h$.

Si en particulier $r = 0$ la fonction $P(x)$ se réduit à zéro. On a donc toujours

$$G_n(x) = 2^n \sum (-1)^{s_1+s_2+\cdots+s_n} \varphi(x+\Omega)$$

quand cette série converge.

Si l'on aime mieux on peut aussi écrire l'équation (13) comme il suit

$$(14)\quad G_n(x) = \lim_{p_1,\ldots,p_n\to\infty} \left[2^n \sum_{s_n=0}^{p_n-1} \cdots \sum_{s_1=0}^{p_1-1} (-1)^{s_1+s_2+\cdots+s_n} \varphi(x+\Omega) \right.$$

$$\left. - \sum_{s_n=0}^{s_n=1} \cdots \sum_{s_1=0}^{s_1=1}{}' (-1)^{\sigma} P(x + s_1 p_1 \omega_1 + \cdots + s_n p_n \omega_n) \right]$$

où

$$\sigma = s_1(p_1-1) + s_2(p_2-1) + \cdots + s_n(p_n-1).$$

Le crochet ' signifie que le cas $s_1 = s_2 = \cdots = s_n = 0$ est à exclure. Dans cette dernière équation $p_1, p_2, \cdots, p_n$ sont des entiers positifs quelconques.

Dans le cas particulier où $\omega_1 = \omega_2 = \cdots = \omega_n = \omega$ notre équation peut se réduire. Si l'on suppose en outre que $p_1 = p_2 = \cdots = p_n = p$, p étant un entier positif quelconque, on trouve

$$G_n(x) = \lim_{p\to\infty}\Big[2^n \sum_{s_n=0}^{p-1} \cdots \sum_{s_1=0}^{p-1} (-1)^{s_1+s_2+\cdots+s_n} \varphi(x+\Omega) - \sum_{s=1}^{s=n} (-1)^{s(p-1)} \binom{n}{s} P(x+sp\omega)\Big].$$

Passons à la fonction F_n. De l'équation

$$\overset{n}{\underset{\omega_1\cdots\omega_n}{\triangle}} F_n(x) = \varphi(x)$$

on déduit la relation identique suivante

$$0 = \sum_{s_n=0}^{p_n-1} \cdots \sum_{s_2=0}^{p_2-1} \sum_{s_1=0}^{p_1-1} \varphi(x+\Omega) - p_1 p_2 \cdots p_n \overset{n}{\underset{p_1\omega_1\cdots p_n\omega_n}{\triangle}} F_n(x),$$

$p_1, p_2, \ldots, p_n$ étant des entiers positifs quelconques. En ajoutant et en soustrayant l'expression

$$Q(x) - (-1)^n p_1 p_2 \cdots p_n \omega_1 \omega_2 \cdots \omega_n \overset{n}{\underset{p_1\omega_1\cdots p_n\omega_n}{\triangle}} Q(x)$$

dans le second membre, cette relation peut s'écrire comme il suit:

$$F_n(x) = (-1)^n \omega_1 \omega_2 \cdots \omega_n \sum_{s_n=0}^{p_n-1} \cdots \sum_{s_1=0}^{p_1-1} \varphi(x+\Omega) + Q(x) - (-1)^n p_1 p_2 \cdots p_n \omega_1 \omega_2 \cdots \omega_n \overset{n}{\underset{p_i\omega_i}{\triangle}} Q(x)$$

$$+\Big[F_n(x) - Q(x) - (-1)^n p_1 p_2 \cdots p_n \omega_1 \omega_2 \cdots \omega_n \overset{n}{\underset{p_i\omega_i}{\triangle}} [F_n(x) - Q(x)]\Big].$$

Faisons tendre tous les p_i vers l'infini. Le dernier terme au second membre tend vers zéro en vertu de l'équation (8). On a donc

$$(15)\quad F_n(x) = \lim_{p_1,\dots,p_n\to\infty}\Big[(-1)^n\omega_1\omega_2\cdots\omega_n\sum_{s_n=0}^{p_n-1}\cdots\sum_{s_1=0}^{p_1-1}\varphi(x+\Omega) + Q(x) - (-1)^n p_1\cdots p_n\,\omega_1\cdots\omega_n\overset{n}{\underset{p_i\omega_i}{\triangle}}Q(x)\Big].$$

Cette équation peut aussi s'écrire comme il suit:

$$(16)\quad F_n(x) = \lim_{p_1,\dots,p_n\to\infty}\Big[(-1)^n\omega_1\omega_2\cdots\omega_n\sum_{s_n=0}^{p_n-1}\cdots\sum_{s_1=0}^{p_1-1}\varphi(x+\Omega) - \sum_{s_n=0}^{s_n=1}\cdots\sum_{s_1=0}^{s_1=1}{}'(-1)^{s_1+s_2+\cdots+s_n}Q(x+s_1p_1\omega_1+\cdots+s_np_n\omega_n)\Big]$$

où le crochet ' signifie que le cas $s_1=s_2=\cdots=s_n=0$ sera exclu.

Dans le cas particulier où $\omega_1=\omega_2=\cdots=\omega_n=\omega$ et $p_1=p_2=\cdots=p_n=p$ notre équation se réduit et prend la forme suivante

$$F_n(x) = \lim_{p\to\infty}\Big[(-1)^n\omega^n\sum_{s_n=0}^{p-1}\cdots\sum_{s_1=0}^{p-1}\varphi(x+\Omega) - \sum_{s=1}^{s=n}(-1)^s\binom{n}{s}Q(x+sp\omega)\Big].$$

Ces égalités ont donc lieu uniformément dans l'intervalle $x\geqq b+h$.

20. On peut aussi exprimer G_n et F_n par G_{n-1} et F_{n-1}. En effet des équations

$$\underset{\omega_n}{\nabla}G_n(x\,|\,\omega_1,\dots,\omega_n) = G_{n-1}(x\,|\,\omega_1,\dots,\omega_{n-1}),$$

$$\underset{\omega_n}{\triangle}F_n(x\,|\,\omega_1,\dots,\omega_n) = F_{n-1}(x\,|\,\omega_1,\dots,\omega_{n-1}),$$

il résulte qu'on a, pour toute valeur entière et positive de p,

$$G_n(x) = 2\sum_{s=0}^{p-1}(-1)^sG_{n-1}(x+s\omega_n)+(-1)^pG_n(x+p\omega_n)$$

$$= 2\sum_{s=0}^{p-1}(-1)^sG_{n-1}(x+s\omega_n)+(-1)^pP(x+p\omega_n) + (-1)^p[G_n(x+p\omega_n)-P(x+p\omega_n)],$$

$$F_n(x) = F_n(x+p\omega_n) - \omega_n \sum_{s=0}^{p-1} F_{n-1}(x+s\omega_n)$$

$$= Q(x+p\omega_n) - \omega_n \sum_{s=0}^{p-1} F_{n-1}(x+s\omega_n) + [F_n(x+p\omega_n) - Q(x+p\omega_n)].$$

Quand p augmente indéfiniment il vient

$$G_n(x \mid \omega_1, \ldots, \omega_n) = \lim_{p\to\infty} \left[2\sum_{s=0}^{p-1} (-1)^s G_{n-1}(x+s\omega_n) + (-1)^p P(x+p\omega_n) \right],$$

$$F_n(x \mid \omega_1, \ldots, \omega_n) = \lim_{p\to\infty} \left[Q(x+p\omega_n) - \omega_n \sum_{s=0}^{p-1} F_{n-1}(x+s\omega_n \mid \omega_1, \ldots, \omega_{n-1}) \right].$$

Dans le cas particulier où

$$\lim_{x\to\infty} \varphi(x) = 0$$

ces équations se réduisent et s'écrivent comme il suit:

$$G_n(x \mid \omega_1, \ldots, \omega_n) = 2\sum_{s=0}^{\infty} (-1)^s G_{n-1}(x+s\omega_n \mid \omega_1, \ldots, \omega_{n-1}),$$

$$F_n(x \mid \omega_1, \ldots, \omega_n) = \lim_{p\to\infty} \left[\int_a^{x+p\omega_n} \frac{B_{n-1}^{(n)}(x+p\omega_n - z)}{(n-1)!} \varphi(z)\, dz - \omega_n \sum_{s=0}^{p-1} F_{n-1}(x+s\omega_n) \right].$$

21. Nous avons défini les fonctions $G_n(x)$ et $F_n(x)$ en partant des séries généralement divergentes

$$2^n \sum (-1)^{s_1+s_2+\cdots+s_n} \varphi(x+\Omega), \tag{17}$$

$$\omega_1 \cdots \omega_n \sum \varphi(x+\Omega). \tag{18}$$

Je veux maintenant démontrer qu'on peut représenter ces fonctions par des séries convergentes. Par voie d'induction on démontre les relations identiques suivantes

$$2^n \sum_{s_n=0}^{p_n-1} \cdots \sum_{s_2=0}^{p_2-1} \sum_{s_1=0}^{p_1-1} (-1)^{s_1+s_2+\cdots+s_n} \overset{n}{\underset{\omega_1 \cdots \omega_n}{\nabla}} P(x+\Omega)$$

$$= \sum_{s_n=0}^{s_n=1} \cdots \sum_{s_2=0}^{s_2=1} \sum_{s_1=0}^{s_1=1} (-1)^{\sigma} P(x+s_1 p_1 \omega_1 + s_2 p_2 \omega_2 + \cdots + s_n p_n \omega_n),$$

$$\sum_{s_n=0}^{p_n-1} \cdots \sum_{s_2=0}^{p_2-1} \sum_{s_1=0}^{p_1-1} \overset{n}{\underset{\omega_1 \cdots \omega_n}{\triangle}} Q(x+\Omega) = p_1 p_2 \cdots p_n \overset{n}{\underset{p_1\omega_1 \cdots p_n\omega_n}{\triangle}} Q(x).$$

Puisque les expressions (14) et (15) convergent uniformément vers les fonctions G_n et F_n on en conclut que ces fonctions se représentent par les séries à n entrées

$$(19) \quad G_n(x) = P(x) + 2^n \sum (-1)^{s_1+s_2+\cdots+s_n} \left[\varphi(x+\Omega) - \overset{n}{\underset{\omega_1 \cdots \omega_n}{\nabla}} P(x+\Omega) \right],$$

$$(20) \quad F_n(x) = Q(x) + (-1)^n \omega_1 \omega_2 \cdots \omega_n \sum \left[\varphi(x+\Omega) - \overset{n}{\underset{\omega_1 \cdots \omega_n}{\triangle}} Q(x+\Omega) \right],$$

les sommations étant étendues à toutes les valeurs entières, non négatives de s_1, de $s_2, \ldots$ et de s_n. Ces séries convergent donc uniformément dans l'intervalle $x \geqq b+h$. Les séries se réduisent quand on pose

$$h = \frac{\omega_1 + \omega_2 + \cdots + \omega_n}{2}.$$

Car en ce cas le nombre des termes qui figurent dans les expressions $P(x)$ et $Q(x)$ est seulement la moitié de ce qu'il est dans le cas général.

De l'analyse du paragraphe 19 il résulte que si l'on fait tendre d'abord p_1, puis $p_2, \ldots$ et enfin p_n vers l'infini les expressions (14) et (15) tendent encore vers des limites et vers les mêmes limites que dans le cas où ces nombres tendent simultanément vers l'infini. On en conclut que les fonctions G_n et F_n se répresentent encore par les séries n fois itérées

$$G_n(x) = P(x) + 2^n \sum_{s_n=0}^{\infty} (-1)^{s_n} \cdots \sum_{s_2=0}^{\infty} (-1)^{s_2} \sum_{s_1=0}^{\infty} (-1)^{s_1} \left[\varphi(x+\Omega) - \overset{n}{\underset{\omega_i}{\nabla}} P(x+\Omega) \right],$$

$$F_n(x) = Q(x) + (-1)^n \omega_1 \omega_2 \cdots \omega_n \sum_{s_n=0}^{\infty} \cdots \sum_{s_2=0}^{\infty} \sum_{s_1=0}^{\infty} \left[\varphi(x+\Omega) - \overset{n}{\underset{\omega_i}{\triangle}} Q(x+\Omega) \right].$$

Il convient de remarquer que les séries, dont nous venons de démontrer la convergence uniforme, en général ne sont pas absolument convergentes. Nous avons choisi l'entier r aussi petit que possible pour donner aux séries la forme la plus simple, mais en augmentant la valeur de r dans les expressions $P(x)$ et $Q(x)$ on peut souvent augmenter la rapidité de la convergence et pour une certaine valeur de r on arrive à une série absolument convergente. Supposons, pour fixer les idées, qu'il existe un entier m_1 tel que

$$\lim_{x\to\infty} x^{2n+\varepsilon}\varphi^{(m_1+1)}(x) = 0, \qquad m_1 \geqq n,$$

et soit m le plus petit entier tel que l'équation (1) soit satisfaite. Remplaçons r par $m+1$ dans l'expression $P(x)$ et par $m+1-n$ dans l'expression $Q(x)$. On a alors

$$\varphi(x) - \overset{n}{\underset{\omega_i}{\nabla}} P(x) = \sum_{\nu=m+1}^{m_1} \frac{E_\nu^{(n)}(h)}{\nu!} \overset{n}{\underset{\omega_i}{\nabla}} \varphi^{(\nu)}(x-h) + \int_{-h}^{\infty} \frac{\bar{E}_{m_1}^{(n)}(-t)}{m_1!} \overset{n}{\underset{\omega_i}{\nabla}} \varphi^{(m_1+1)}(x+t)\,dt,$$

$$\varphi(x) - \overset{n}{\underset{\omega_i}{\triangle}} Q(x) = \sum_{\nu=m+1}^{m_1} \frac{B_\nu^{(n)}(h)}{\nu!} \overset{n}{\underset{\omega_i}{\triangle}} f^{(\nu)}(x-h) + \int_{-h}^{\infty} \frac{\bar{B}_{m_1}^{(n)}(-t)}{m_1!} \overset{n}{\underset{\omega_i}{\triangle}} f^{(m_1+1)}(x+t)\,dt.$$

On voit sur ces équations que le terme général des séries (19) et (20) est en valeur absolue plus petit que

$$\frac{C}{(x+\Omega)^{n+\varepsilon}},$$

C étant une constante. La série (19) converge donc absolument si l'on remplace r par $m+1$ et la série (20) converge absolument si l'on remplace r par $m+1-n$. Il y a pourtant une différence à noter entre les deux séries. Pour la série (19), la valeur de r qui entraîne la convergence absolue est en général plus grande que celle qui entraîne la convergence simple. Mais pour la série (20) il arrive bien souvent que ces deux valeurs coïncident. Pour élucider ce fait considérons les trois exemples suivants. Soit d'abord

$$\varphi(x) = \frac{1}{x}.$$

L'équation (2) est satisfaite pour $r = 0$. On a donc $P(x) = 0$ et la série (17) converge, mais elle n'est pas absolument convergente. Pour que la série (19)

converge absolument il faut et il suffit que $r \geqq n$. La série (18) diverge mais la série (20) converge absolument, si $r = 0$, et *a fortiori*, si $r > 0$.

Supposons, en second lieu, que

$$\varphi(x) = x^{\beta},$$

β étant un nombre positif qui n'est pas un entier. La série (17) diverge. La série (19) converge simplement si $\beta < r < \beta + n$. Cette série sera absolument convergente si $r > \beta + n$. La série (20) sera absolument convergente, si $r > \beta$, et elle diverge si $r < \beta$. Dans ces deux cas toute série de la forme (20) qui converge est en même temps absolument convergente.

Soit enfin

$$\varphi(x) = \sin(x^{\alpha}), \qquad (0 < \alpha < 1).$$

Les séries (17) et (18) divergent. La série (19) converge pour $r \geqq 1$ et elle converge absolument pour $r > \dfrac{n}{1-\alpha}$. La série (20) sera convergente si $r \geqq 1$. Mais pour que cette série converge absolument il faut en outre que $r > \dfrac{\alpha n}{1-\alpha}$, et ce nombre est supérieur à 1, si $\alpha > \dfrac{1}{n+1}$.

22. Reprenons les développements

$$(21) \qquad G_n(x+h \mid \omega_1, \omega_2, \ldots, \omega_n) = \sum_{\nu=0}^{m} \frac{E_{\nu}^{(n)}(h \mid \omega_1, \ldots, \omega_n)}{\nu!} \varphi^{(\nu)}(x)$$

$$+ \int_0^{\infty} \frac{\bar{E}_m^{(n)}(h-t)}{m!} \varphi^{(m+1)}(x+t)\, dt,$$

$$(22) \qquad F_n(x+h \mid \omega_1, \omega_2, \ldots, \omega_n) = \sum_{\nu=0}^{n+m} \frac{B_{\nu}^{(n)}(h \mid \omega_1, \ldots, \omega_n)}{\nu!} f^{(\nu)}(x)$$

$$+ \int_0^{\infty} \frac{\bar{B}_{m+n}^{(n)}(h-t)}{(m+n)!} \varphi^{(m+1)}(x+t)\, dt,$$

où l'on a posé

$$f(x) = \int_a^x \frac{(x-z)^{n-1}}{(n-1)!} \varphi(z)\, dz.$$

Nous avons vu que ces deux séries mettent en évidence la manière dont se comportent les fonctions G_n et F_n pour les valeurs très grandes de la variable x. Laissons maintenant x fixe et considérons les ω_i comme des variables. Le point $\omega_1 = \omega_2 = \cdots = \omega_n = 0$ est, en général, un point singulier* des fonctions G_n et F_n, mais les deux séries nous donnent des renseignements sur la manière dont se comportent les fonctions G_n et F_n pour les valeurs positives et très petites des variables $\omega_1, \omega_2, \ldots, \omega_n$. En effet, remplaçons h par λh et ω_i par $\lambda\omega_i$, on obtient deux séries procédant suivant les puissances croissantes de λ car on a, quel que soit λ,

$$E_\nu^{(n)}(\lambda h \mid \lambda\omega_1, \lambda\omega_2, \ldots, \lambda\omega_n) = \lambda^\nu E_\nu^{(n)}(h \mid \omega_1, \omega_2, \ldots, \omega_n),$$

$$B_\nu^{(n)}(\lambda h \mid \lambda\omega_1, \lambda\omega_2, \ldots, \lambda\omega_n) = \lambda^\nu B_\nu^{(n)}(h \mid \omega_1, \omega_2, \ldots, \omega_n).$$

On aura donc

$$(23)\quad G_n(x+\lambda h \mid \lambda\omega_1, \lambda\omega_2, \ldots, \lambda\omega_n) = \sum_{\nu=0}^{m} \frac{\lambda^\nu}{\nu!} E_\nu^{(n)}(h \mid \omega_1, \omega_2, \ldots, \omega_n)\, \varphi^{(\nu)}(x) + \mathfrak{R}_m,$$

$$(24)\quad F_n(x+\lambda h \mid \lambda\omega_1, \lambda\omega_2, \ldots, \lambda\omega_n) = \sum_{\nu=0}^{n+m} \frac{\lambda^\nu}{\nu!} B_\nu^{(n)}(h \mid \omega_1, \omega_2, \ldots, \omega_n)\, f^{(\nu)}(x) + R_{m+n},$$

où

$$(25)\quad \mathfrak{R}_m = \lambda^{m+1} \int_0^\infty \frac{\overline{E}_m^{(n)}(h-z)}{m!}\, \varphi^{(m+1)}(x+\lambda z)\, dz,$$

et

$$R_{m+n} = \lambda^{m+n+1} \int_0^\infty \frac{\overline{B}_{m+n}^{(n)}(h-z)}{(m+n)!}\, \varphi^{(m+1)}(x+\lambda z)\, dz.$$

Cherchons une limite supérieure du reste en supposant λ positive et très petite. On sait trouver une constante C telle que

$$|R_{m+n}| < \lambda^{m+n+1} C \int_0^\infty \frac{z^{n-1} dz}{(x+\lambda z)^{n+\epsilon}} = \lambda^{m+1} \frac{C}{x^\epsilon} \int_0^\infty \frac{z^{n-1} dz}{(1+z)^{n+\epsilon}}.$$

* Cf. Acta Mathematica, vol. 44, pp. 178–198.

La fonction $| R_{m+n}\lambda^{-m-1} |$ reste donc plus petite qu'une constante, quand λ tend vers zéro pendant que x reste fixe. Cela posé, remarquons qu'on a

$$R_p = \sum_{\nu=p+1}^{m+n} \frac{\lambda^\nu}{\nu!} B_\nu^{(m)}(h)\, f^{(\nu)}(x) + R_{m+n}.$$

En supposant $p \leqq m$ on en conclut que

$$\lim_{\lambda \to 0} R_p \lambda^{-p} = 0. \tag{26}$$

En considérant l'expression (25) on démontre de même que

$$\lim_{\lambda \to 0} \mathfrak{R}_p \lambda^{-p} = 0, \tag{27}$$

pourvu que $p \leqq m-n$. Si l'on suppose que la fonction $\varphi(x)$ admet, pour $x \geqq b$, des dérivées de tous les ordres et que

$$\lim_{x \to \infty} x^{n+\varepsilon} \varphi^{(m)}(x) = 0$$

pour toute valeur de m qui surpasse un certain nombre alors les équations (26) et (27) sont vraies quel que soit l'entier positif p. Par conséquent les séries (23) et (24) représentent les fonctions au premier membre asymptotiquement, au sens de Poincaré, pour les valeurs positives et très petites de λ. On a donc en particulier

$$\lim_{\lambda \to 0} G_n(x \mid \lambda\omega_1, \lambda\omega_2, \ldots, \lambda\omega_n) = \varphi(x), \tag{28}$$

$$\lim_{\lambda \to 0} F_n(x \mid \lambda\omega_1, \lambda\omega_2, \ldots, \lambda\omega_n) = \int_a^x \frac{(x-z)^{n-1}}{(n-1)!} \varphi(z)\, dz. \tag{29}$$

Nos deux séries sont encore asymptotiques dans un autre sens. On peut considérer les séries (21) et (22) comme des séries procédant suivant les puissances de ω_1, de $\omega_2, \ldots,$ de ω_n et dans lesquelles les termes ont été groupés d'une certaine manière. Mais on peut ranger autrement les termes par exemple suivant les puissances d'un seul des ω, soit ω_n. En groupant

ainsi les termes quel sens aura alors la série? Il est facile de le voir car de l'équation

$$\underset{\omega_n}{\nabla} G_n(x \mid \omega_1, \ldots, \omega_n) = G_{n-1}(x \mid \omega_1, \ldots, \omega_{n-1})$$

on tire le développement

$$G_n(x \mid \omega_1, \omega_2, \ldots, \omega_n) = \sum_{\nu=0}^{m} \frac{\omega_n^\nu C_\nu}{\nu!\, 2^\nu} \frac{d^\nu G_{n-1}(x \mid \omega_1, \ldots, \omega_{n-1})}{dx^\nu} + R.$$

En considérant x et $\omega_1, \omega_2, \ldots, \omega_{n-1}$ comme des nombres fixes et ω_n comme une variable positive il résulte de ce que nous venons de dire que cette série représente la fonction G_n asymptotiquement au voisinage du point $\omega_n = 0$. On aura donc en particulier

$$\lim_{\omega_n \to 0} G_n(x \mid \omega_1, \omega_2, \ldots, \omega_n) = G_{n-1}(x \mid \omega_1, \omega_2, \ldots, \omega_{n-1}). \tag{30}$$

IV.

Quelques propriétés des solutions principales

23. Posons

$$G_n(x \mid \omega_1, \omega_2, \ldots, \omega_n; \eta) = 2^n \sum (-1)^{s_1+s_2+\cdots+s_n} \varphi(x+\Omega) e^{-\eta(x+\Omega)},$$

$$F_n(x \mid \omega_1, \omega_2, \ldots, \omega_n; \eta) = \int_a^\infty \frac{B_{n-1}^{(n)}(x-z)}{(n-1)!} \varphi(z) e^{-\eta z} dz + (-1)^n \omega_1 \omega_2 \cdots \omega_n \sum \varphi(x+\Omega) e^{-\eta(x+\Omega)}.$$

Nous avons défini les fonctions G_n et F_n par les équations*

$$G_n(x \mid \omega_1, \ldots, \omega_n) = \lim_{\eta \to 0} G_n(x \mid \omega_1, \ldots, \omega_n; \eta), \tag{1}$$

$$F_n(x \mid \omega_1, \ldots, \omega_n) = \lim_{\eta \to 0} F_n(x \mid \omega_1, \ldots, \omega_n; \eta). \tag{2}$$

*Nous supposerons que la fonction $\varphi(x)$ satisfait à la condition du paragraphe 8 ce qui entraîne l'existence des limites (1) et (2). Dans un autre mémoire, que nous publierons plus tard, nous démontrerons l'existence de ces deux limites en faisant d'autres hypothèses relativement à la fonction $\varphi(x)$. Tous les théorèmes que nous allons établir restent vrais dans ces cas et la plupart des démonstrations s'appliquent sans modification.

La fonction $G_n(x)$ se nomme *la somme de première espece de la fonction* $\varphi(x)$. Pour abréger l'écriture nous représenterons souvent cette fonction par la notation

$$\sum \varphi(x) \overset{n}{\underset{\omega_1 \dots \omega_n}{\nabla}} x = G_n(x|\omega_1, \omega_2, \dots, \omega_n).$$

La fonction $F_n(x)$ se nomme *le somme de seconde espèce de la fonction* $\varphi(x)$ et se représente par la notation

$$\int_a^x \varphi(z) \overset{n}{\underset{\omega_1 \dots \omega_n}{\Delta}} z = F_n(x|\omega_1, \omega_2, \dots, \omega_n).$$

On aura donc pour la somme de première espèce

$$\overset{n}{\underset{\omega_1 \dots \omega_n}{\nabla}} \sum \varphi(x) \overset{n}{\underset{\omega_1 \dots \omega_n}{\nabla}} x = \varphi(x) \tag{3}$$

et pour la somme de seconde espèce

$$\overset{n}{\underset{\omega_1 \dots \omega_n}{\Delta}} \int_a^x \varphi(z) \overset{n}{\underset{\omega_1 \dots \omega_n}{\Delta}} z = \varphi(x). \tag{4}$$

Les deux opérations de sommation sont donc, pour ainsi dire, les opérations inverses aux opérations $\overset{n}{\nabla}$ et $\overset{n}{\Delta}$. Je vais indiquer quelques propriétés remarquables des sommes qui résultent presque immédiatement de leur définition. *Un facteur constant c peut être mis hors du signe de sommation,* c'est-à-dire que

$$\sum c\varphi(x) \overset{n}{\underset{\omega_1 \dots \omega_n}{\nabla}} x = c \sum \varphi(x) \overset{n}{\underset{\omega_1 \dots \omega_n}{\nabla}} x,$$

$$\int_a^x c\varphi(z) \overset{n}{\underset{\omega_1 \dots \omega_n}{\Delta}} z = c \int_a^x \varphi(z) \overset{n}{\underset{\omega_1 \dots \omega_n}{\Delta}} z.$$

La somme de $\varphi_1(x)$ *augmentée de la somme de* $\varphi_2(x)$ *est égale à la somme de* $\varphi_1(x) + \varphi_2(x)$, c'est-à-dire que

$$\int \varphi_1(x) \overset{n}{\underset{\omega_1 \dots \omega_n}{\nabla}} x + \int \varphi_2(x) \overset{n}{\underset{\omega_1 \dots \omega_n}{\nabla}} x = \int (\varphi_1(x) + \varphi_2(x)) \overset{n}{\underset{\omega_1 \dots \omega_n}{\nabla}} x,$$

$$\int_a^x \varphi_1(z) \overset{n}{\underset{\omega_1 \dots \omega_n}{\Delta}} z + \int_a^x \varphi_2(z) \overset{n}{\underset{\omega_1 \dots \omega_n}{\Delta}} z = \int_a^x (\varphi_1(z) + \varphi_2(z)) \overset{n}{\underset{\omega_1 \dots \omega_n}{\Delta}} z.$$

Soient $m_1, m_2, \dots, m_n$ des entiers positifs quelconques. Le polynome de Bernoulli satisfait à la relation*

$$\sum_{s_1=0}^{m_1-1} B_\nu^{(n)}\left(x + \frac{s_1 \omega_1}{m_1} \mid \omega_1, \omega_2, \dots, \omega_n\right) = m_1 B_\nu^{(n)}\left(x \mid \frac{\omega_1}{m_1}, \omega_2, \dots, \omega_n\right).$$

On aura donc pour toute valeur positive de η

$$\sum_{s_1=0}^{m_1-1} F_n\left(x + \frac{s_1 \omega_1}{m_1} \mid \omega_1, \omega_2, \dots, \omega_n; \eta\right) = m_1 F_n\left(x \mid \frac{\omega_1}{m_1}, \omega_2, \dots, \omega_n; \eta\right)$$

parce que dans une série absolument convergente on peut ranger les termes dans tel ordre que l'on veut. Faisons tendre le nombre positif η vers zéro, il vient

$$(5) \quad \sum_{s_1=0}^{m_1-1} F_n\left(x + \frac{s_1 \omega_1}{m_1} \mid \omega_1, \omega_2, \dots, \omega_n\right) = m_1 F_n\left(x \mid \frac{\omega_1}{m_1}, \omega_2, \dots, \omega_n\right).$$

Puisque F_n est une fonction symétrique des variables $\omega_1, \omega_2, \dots, \omega_n$ on en conclut que

$$(6) \quad \sum_{s_n=0}^{m_n-1} \cdots \sum_{s_2=0}^{m_2-1} \sum_{s_1=0}^{m_1-1} F_n\left(x + \frac{s_1 \omega_1}{m_1} + \frac{s_2 \omega_2}{m_2} + \cdots + \frac{s_n \omega_n}{m_n} \mid \omega_1, \omega_2, \dots, \omega_n\right)$$

$$= m_1 m_2 \cdots m_n F_n\left(x \mid \frac{\omega_1}{m_1}, \frac{\omega_2}{m_2}, \dots, \frac{\omega_n}{m_n}\right).$$

* Acta Mathematica, vol. 43, p. 169.

Soient maintenant $m_1, m_2, \ldots, m_n$ des entiers positifs pairs. On vérifie aisément l'identité suivante

$$\sum_{s_n=0}^{m_n-1} \cdots \sum_{s_2=0}^{m_2-1} \sum_{s_1=0}^{m_1-1} (-1)^{s_1+s_2+\cdots+s_n} f(x+s_1\omega_1+s_2\omega_2+\cdots+s_n\omega_n)$$

$$= (-1)^n \omega_1\omega_2\cdots\omega_n \sum_{s_n=0}^{\frac{1}{2}m_n-1} \cdots \sum_{s_2=0}^{\frac{1}{2}m_2-1} \sum_{s_1=0}^{\frac{1}{2}m_1-1} \underset{\omega_1\ldots\omega_n}{\overset{n}{\triangle}} f(x+2s_1\omega_1+2s_2\omega_2+\cdots+2s_n\omega_n).$$

Si en particulier $f(x)$ est un polynome d'un degré inférieur à n, chaque terme dans le second membre s'annule. Le polynome de Bernoulli $B_{n-1}^{(n)}(x)$ est du degré $n-1$. Ce polynome satisfait donc à la relation

$$\sum_{s_n=0}^{m_n-1} \cdots \sum_{s_2=0}^{m_2-1} \sum_{s_1=0}^{m_1-1} (-1)^{s_1+s_2+\cdots+s_n} B_{n-1}^{(n)}\left(x+\frac{s_1\omega_1}{m_1}+\frac{s_2\omega_2}{m_2}\cdots+\frac{s_n\omega_n}{m_n}\right)=0.$$

On aura par conséquent pour toute valeur positive de η

$$\sum_{s_n=0}^{m_n-1} \cdots \sum_{s_1=0}^{m_1-1} (-1)^{s_1+\cdots+s_n} F_n\left(x+\frac{s_1\omega_1}{m_1}+\cdots+\frac{s_n\omega_n}{m_n} \mid \omega_1, \ldots, \omega_n; \eta\right)$$

$$= \left(-\frac{1}{2}\right)^n \omega_1\omega_2\cdots\omega_n G_n\left(x \mid \frac{\omega_1}{m_1}, \frac{\omega_2}{m_2}, \ldots, \frac{\omega_n}{m_n}; \eta\right)$$

parce que les séries qui entrent en considération convergent absolument. Faisons tendre le nombre positif η vers zéro, il vient

$$(7) \quad \sum_{s_n=0}^{m_n=1} \cdots \sum_{s_1=0}^{m_1-1} (-1)^{s_1+\cdots+s_n} F_n\left(x+\frac{s_1\omega_1}{m_1}+\cdots+\frac{s_n\omega_n}{m_n} \mid \omega_1, \ldots, \omega_n\right)$$

$$= \left(-\frac{1}{2}\right)^n \omega_1\ \omega_2\cdots\omega_n\ G_n\left(x \mid \frac{\omega_1}{m_1}, \frac{\omega_2}{m_2}, \ldots, \frac{\omega_n}{m_n}\right).$$

Dans le cas particulier où $m_1 = m_2 = \cdots = m_n = 2$ cette équation peut s'écrire sous la forme suivante

$$(7\text{ bis}) \quad G_n(x \mid \omega_1, \omega_2, \ldots, \omega_n) = \overset{n}{\underset{\omega_1 \ldots \omega_n}{\triangle}} F_n(x \mid 2\omega_1, 2\omega_2, \ldots, 2\omega_n),$$

et de même l'équation (6) prend la forme

$$(6\text{ bis}) \quad F_n(x \mid \omega_1, \omega_2, \ldots, \omega_n) = \overset{n}{\underset{\omega_1 \ldots \omega_n}{\nabla}} F_n(x \mid 2\omega_1, 2\omega_2, \ldots, 2\omega_n).$$

Avec la notation que nous venons d'introduire ces relations entre les sommes de première et de seconde espèce s'écrivent comme il suit,

$$(7\text{ter}) \quad \int q(x) \overset{n}{\underset{\omega_1 \ldots \omega_n}{\nabla}} x = \overset{n}{\underset{\omega_1 \ldots \omega_n}{\triangle}} \int_a^x q(x) \overset{n}{\underset{2\omega_1 \ldots 2\omega_n}{\triangle}} x,$$

$$(6\text{ter}) \quad \int_a^x q(x) \overset{n}{\underset{\omega_1 \ldots \omega_n}{\triangle}} x = \overset{n}{\underset{\omega_1 \ldots \omega_n}{\nabla}} \int_a^x q(x) \underset{2\omega_1 \ldots 2\omega_n}{\triangle} x.$$

De même on aura en vertu de l'équation (5)

$$(5\text{bis}) \quad F_n(x \mid \omega_1, \omega_2, \ldots, \omega_n) = \underset{\omega_1}{\nabla} F_n(x \mid 2\omega_1, \omega_2, \ldots, \omega_n).$$

Soient en dernier lieu $m_1, m_2, \ldots, m_n$ des entiers positifs impairs. On aura, si $\eta > 0$,

$$\sum_{s_1=0}^{m_1-1} (-1)^{s_1} G_n\left(x + \frac{s_1\omega_1}{m_1} \mid \omega_1, \omega_2, \ldots, \omega_n; \eta\right) = G_n\left(x \mid \frac{\omega_1}{m_1}, \omega_2, \ldots, \omega_n; \eta\right).$$

En faisant tendre η vers zéro on trouvera

$$(8) \quad \sum_{s_1=0}^{m_1-1} (-1)^{s_1} G_n\left(x + \frac{s_1\omega_1}{m_1} \mid \omega_1, \omega_2, \ldots, \omega_n\right) = G_n\left(x \mid \frac{\omega_1}{m_1}, \omega_2, \ldots, \omega_n\right).$$

On en déduit la relation plus générale

$$(9)\qquad \sum_{s_n=0}^{m_n-1}\cdots\sum_{s_1=0}^{m_1-1}(-1)^{s_1+\cdots+s_n}\,G_n\left(x+\frac{s_1\omega_1}{m_1}+\cdots+\frac{s_n\omega_n}{m_n}\,|\,\omega_1,\ \ldots,\ \omega_n\right)$$

$$=G_n\left(x\,|\,\frac{\omega_1}{m_1},\ \frac{\omega_2}{m_2},\ \ldots,\ \frac{\omega_n}{m_n}\right).$$

24. Ces formules se prêtent à un passage à la limite. Soit Θ_s un nombre tel que $0<\Theta_s<1$. L'équation (8) peut s'écrire comme il suit

$$G_n(x)+\frac{\omega_1}{m_1}\sum_{s=1}^{\frac{1}{2}(m_1-1)}\frac{d}{dx}\,G_n\left(x+\frac{(2s-1)\,\omega_1}{m_1}+\frac{\Theta_s\,\omega_1}{m_1}\right)=G_n\left(x\,|\,\frac{\omega_1}{m_1},\,\omega_2,\,\ldots,\omega_n\right).$$

Cette équation est vraie quelque grand que soit l'entier m_1. Quand m_1 augmente indéfiniment le premier membre tend vers la limite

$$G_n(x\,|\,\omega_1,\,\omega_2,\,\ldots,\,\omega_n)+\tfrac{1}{2}\int_x^{x+\omega_1}\frac{d}{dx}\,G_n(x\,|\,\omega_1,\,\omega_2,\,\ldots,\,\omega_n)\,dx$$

$$=G_n(x)+\tfrac{1}{2}\,G_n(x+\omega_1)-\tfrac{1}{2}\,G_n(x)$$

$$=G_{n-1}(x\,|\,\omega_2,\,\omega_3,\,\ldots,\,\omega_n).$$

On aura donc

$$(10)\qquad \lim_{m_1\to\infty}G_n\left(x\,|\,\frac{\omega_1}{m_1},\,\omega_2,\,\ldots,\,\omega_n\right)=G_{n-1}(x\,|\,\omega_2,\,\omega_3,\,\ldots,\,\omega_n).$$

On en conclut sans peine que

$$(11)\qquad \lim_{\omega_{p+1},\,\omega_{p+2},\,\ldots,\,\omega_n\to 0}G_n(x|\,\omega_1,\,\omega_2,\,\ldots,\,\omega_n)=G_p(x|\,\omega_1,\,\omega_2,\,\ldots,\,\omega_p)$$

et que

$$(12)\qquad \lim_{\omega_1,\,\ldots,\,\omega_n\to 0}G_n(x|\,\omega_1,\,\omega_2,\,\ldots,\,\omega_n)=\varphi(x),$$

comme nous l'avons déjà vu dans le paragraphe 22.

Divisons les deux membres de l'équation (5) par m_1 et faisons ensuite tendre l'entier m_1 vers l'infini. Le premier membre tend vers l'intégrale

$$\frac{1}{\omega_1}\int_x^{x+\omega_1} F_n(x)\,dx.$$

Le second membre tendra donc aussi vers une limite et l'on trouvera

$$\frac{1}{\omega_1}\int_x^{x+\omega_1} F_n(x|\omega_1,\omega_2,\ldots,\omega_n)\,dx = \lim_{m_1\to\infty} F_n(x|\frac{\omega_1}{m_1},\omega_2,\ldots,\omega_n). \tag{13}$$

De méme, divisons les deux membres de l'équation (6) par le produit des nombres $m_1, m_2, \ldots, m_n$. Soit p un des nombres $1, 2, \ldots, n$. Posons $m_{p+1} = m_{p+2} = \cdots = m_n = 1$ et admettons que les nombres $m_1, m_2, \ldots, m_p$ augmentent indéfiniment. On trouvera

$$\frac{1}{\omega_1\omega_2\cdots\omega_p}\int_0^{\omega_p} dt_p\cdots\int_0^{\omega_2} dt_2\int_0^{\omega_1} F_n(x+t_1+t_2+\cdots+t_p|\,\omega_1,\omega_2,\ldots,\omega_n)\,dt_1$$

$$= \lim_{m_1\cdots m_p\to\infty} F_n\Big(x|\frac{\omega_1}{m_1},\ldots,\frac{\omega_p}{m_p},\omega_{p+1},\ldots,\omega_n\Big). \tag{14}$$

Si en particulier $p = n$, la limite au second membre est égale à la fonction $f(x)$; c'est ce qui résulte de l'équation (29) paragraphe 22. On aura donc

$$\int_0^1 dt_n\cdots\int_0^1 dt_2\int_0^1 F_n(x+\omega_1 t_1+\omega_2 t_2+\cdots+\omega_n t_n|\,\omega_1,\omega_2,\ldots,\omega_n)\,dt_1$$

$$= \int_a^x \frac{(x-z)^{n-1}}{(n-1)!}\varphi(z)\,dz. \tag{15}$$

25. Je vais donner une démonstration plus directe de cette relation remarquable. Reprenons la série qui entre dans l'expression de $F_n(x;\eta)$ et considérons l'intégrale suivante

$$\int_0^{\omega_n}\cdots\int_0^{\omega_2}\int_0^{\omega_1}\Sigma\,\varphi(x+t_1+t_2+\cdots+t_n+\Omega)\,e^{-\eta(x+t_1+t_2+\cdots+t_n+\Omega)}\,dt_1\,dt_2\cdots dt_n.$$

η étant positif, on peut intégrer terme par terme. Cette intégrale est donc égale à

$$\Sigma\int_0^{\omega_n}\cdots\int_0^{\omega_2}\int_0^{\omega_1}\varphi(x+t_1+t_2+\cdots+t_n+\Omega)\,e^{-\eta(x+t_1+t_2+\cdots+t_n+\Omega)}\,dt_1\,dt_2\cdots dt_n$$

$$=\int_0^{\infty}\cdots\int_0^{\infty}\int_0^{\infty}\varphi(x+t_1+t_2+\cdots+t_n)\,e^{-\eta(x+t_1+t_2+\cdots+t_n)}\,dt_1\,dt_2\cdots dt_n$$

$$=\int_0^{\infty}\frac{t^{n-1}}{(n-1)!}\,\varphi(x+t)\,e^{-\eta(x+t)}\,dt$$

$$=\int_x^{\infty}\frac{(z-x)^{n-1}}{(n-1)!}\,\varphi(z)\,e^{-\eta z}\,dz.$$

Cela posé, remarquons que le polynome de Bernoulli satisfait à la relation*

$$\frac{1}{\omega_1\,\omega_2\cdots\omega_n}\int_0^{\omega_n}\cdots\int_0^{\omega_2}\int_0^{\omega_1}B_\nu^{(n)}(x+t_1+t_2+\cdots+t_n)\,dt_1\,dt_2\cdots dt_n=x^\nu.$$

On aura donc pour toute valeur positive de η

$$\frac{1}{\omega_1\,\omega_2\cdots\omega_n}\int_0^{\omega_n}\cdots\int_0^{\omega_2}\int_0^{\omega_1}F_n(x+t_1+t_2+\cdots+t_n\,|\,\omega_1,\omega_2,\ldots,\omega_n;\eta)\,dt_1\,dt_2\cdots dt_n$$

$$=\int_a^{\infty}\frac{(x-z)^{n-1}}{(n-1)!}\,\varphi(z)\,e^{-\eta z}\,dz-\int_x^{\infty}\frac{(x-z)^{n-1}}{(n-1)!}\,\varphi(z)\,e^{-\eta z}\,dz$$

$$=\int_a^{x}\frac{(x-z)^{n-1}}{(n-1)!}\,\varphi(z)\,e^{-\eta z}\,dz.$$

* Acta Mathematica, vol. 43, p. 172.

Faisons tendre η vers zéro. Puisque la fonction $F_n(x;\eta)$ tend uniformément vers la limite $F_n(x)$ on trouvera

$$\frac{1}{\omega_1\,\omega_2\cdots\omega_n}\int_0^{\omega_n} dt_n \cdots \int_0^{\omega_2} dt_2 \int_0^{\omega_1} F_n(x+t_1+t_2+\cdots+t_n \mid \omega_1, \omega_2, \ldots, \omega_n)\,dt_1$$

$$= \int_a^x \frac{(x-z)^{n-1}}{(n-1)!}\,\varphi(z)\,dz.$$

Cherchons maintenant à évaluer l'intégrale au premier membre de l'équation (13). Je dis qu'on a

$$(16)\qquad \frac{1}{\omega_1}\int_0^{\omega_1} F_n(x+t \mid \omega_1, \omega_2, \ldots, \omega_n)\,dt = \sum_a^x \left(\int_a^z \varphi(z)\,dz\right)^{\overset{n-1}{\triangle}}_{\omega_2\ldots\omega_n} z.$$

Rappelons d'abord que cette équation est vraie dans le cas où F_n se réduit au polynome de Bernoulli* c'est-à-dire que

$$\frac{1}{\omega_1}\int_0^{\omega_1} B_\nu^{(n)}(x+t \mid \omega_1, \omega_2, \ldots, \omega_n)\,dt = B_\nu^{(n-1)}(x \mid \omega_2, \omega_3, \ldots, \omega_n).$$

La fonction F_n se développe de la manière suivante

$$(17)\qquad F_n(x+h) = \sum_{\nu=0}^{n-1} \frac{B_\nu^{(n)}(h)}{\nu!}\int_a^x \frac{(x-z)^{n-1-\nu}}{(n-1-\nu)!}\,\varphi(z)\,dz$$

$$+ \sum_{\nu=0}^{m} \frac{B_{\nu+n}^{(n)}(h)}{(\nu+n)!}\,\varphi^{(\nu)}(x) + \int_0^\infty \frac{\overline{B}_{m+n}^{(n)}(h-t)}{(m+n)!}\,\varphi^{(m+1)}(x+t)\,dt.$$

*Acta Mathematica, vol. 43, p. 171.

Intégrons par rapport à h, on trouvera

$$\frac{1}{\omega_1}\int_0^{\omega_1} F_n(x+h\,|\,\omega_1,\ldots,\omega_n)\,dh = \sum_{\nu=0}^{n-1}\frac{B_\nu^{(n-1)}}{\nu!}\int_a^x \frac{(x-z)^{n-1-\nu}}{(n-1-\nu)!}\,\varphi(z)\,dz$$

(18)

$$+\sum_{\nu=0}^{m}\frac{B_{\nu+n}^{(n-1)}}{(\nu+n)!}\,\varphi^{(\nu)}(x)+\int_0^\infty \frac{\overline{B}_{m+n}^{(n-1)}(-t)}{(m+n)!}\,\varphi^{(m+1)}(x+t)\,dt.$$

D'autre part, on a en vertu de l'équation (17)

$$\int_a^x \psi(z)\underset{\omega_2\ldots\omega_n}{\overset{n-1}{\triangle}} z = \sum_{\nu=0}^{n-2}\frac{B_\nu^{(n-1)}}{\nu!}\int_a^x \frac{(x-z)^{n-2-\nu}}{(n-2-\nu)!}\,\psi(z)\,dz$$

(19)

$$+\sum_{\nu=0}^{m+1}\frac{B_{\nu+n-1}^{(n-1)}}{(\nu+n-1)!}\,\psi^{(\nu)}(x)+\int_0^\infty \frac{\overline{B}_{m+n}^{(n-1)}(-t)}{(m+n)!}\,\psi^{(m+2)}(t)\,dt.$$

Si l'on pose en particulier

$$\psi(x)=\int_a^x \varphi(z)\,dz$$

le second membre de l'équation (19) devient identique au second membre de l'équation (18). L'équation (16) est par conséquent vraie. La valeur moyenne de la fonction $F_n(x\,|\,\omega_1,\omega_2,\ldots,\omega_n)$ dans un intervalle de longueur ω_1 est donc une fonction qui ne dépend pas de ω_1. Cette valeur moyenne est une fonction symétrique des variables $\omega_2,\omega_3,\ldots,\omega_n$. En rapprochant l'équation (16) à l'équation (13) on voit qu'on aura

(20)
$$\lim_{\omega_1\to 0}\int_a^x \varphi(z)\underset{\omega_1\ldots\omega_n}{\overset{n}{\triangle}} z = \int_a^x\left(\int_a^z \varphi(z)\,dz\right)\underset{\omega_2\ldots\omega_n}{\overset{n-1}{\triangle}} z.$$

En intégrant la fonction F_n p fois de suite on trouve, en vertu de l'équation (16),

$$\frac{1}{\omega_1\,\omega_2\cdots\omega_p}\int_0^{\omega_p} dt_p\cdots\int_0^{\omega_2} dt_2\int_0^{\omega_1} F_n(x+t_1+t_2+\cdots+t_p \mid \omega_1, \omega_2, \ldots, \omega_n)\,dt_1$$

$$=\sum_a^x\left(\int_a^z \frac{(z-t)^{p-1}}{(p-1)!}\varphi(t)\,dt\right)\overset{n-p}{\underset{\omega_{p+1}\cdots\omega_n}{\triangle}} z.$$

En rapprochant cette équation à l'équation (14) on voit que, si l'on fait tendre les nombres $\omega_1, \omega_2, \ldots, \omega_p$ vers zéro, on trouvera

$$\lim_{\omega_1\ldots\omega_p\to 0}\sum_a^x \varphi(z)\overset{n}{\underset{\omega_1\ldots\omega_n}{\triangle}} z=\sum_a^x\left(\int_a^z \frac{(z-t)^{p-1}}{(p-1)!}\varphi(t)\,dt\right)\overset{n-p}{\underset{\omega_{p+1}\cdots\omega_n}{\triangle}} z. \tag{21}$$

En particulier si $p=n$, tous les nombres $\omega_1, \omega_2, \ldots, \omega_n$ tendent vers zéro et il vient

$$\lim_{\omega_1\ldots\omega_n\to 0}\sum_a^x \varphi(z)\overset{n}{\underset{\omega_1\ldots\omega_n}{\triangle}} z=\int_a^x \frac{(x-z)^{n-1}}{(n-1)!}\varphi(z)\,dz. \tag{22}$$

26. Supposons que la fonction $\varphi(x)$ admet, pour $x\geqq b$, des dérivées continues de tous les ordres et que

$$\lim_{x\to\infty} x^{m+\varepsilon}\varphi^{(\nu)}(x)=0, \qquad \text{si } \nu>m. \tag{23}$$

Considérons les dérivées des fonctions $G_n(x)$ et $F_n(x)$. Ces deux fonctions admettent les développements

$$G_n(x)=\sum_{\nu=0}^{m}\frac{C_\nu^{(n)}}{2^\nu\,\nu!}\varphi^{(\nu)}(x)+\int_0^\infty \frac{\bar{E}_m^{(n)}(-t)}{m!}\varphi^{(m+1)}(x+t)\,dt, \tag{24}$$

$$F_n(x)=\sum_{\nu=0}^{m+n}\frac{B_\nu^{(n)}}{\nu!}f^{(\nu)}(x)+\int_0^\infty \frac{\bar{B}_{m+n}^{(n)}(-t)}{(m+n)!}\varphi^{(m+1)}(x+t)\,dt, \tag{25}$$

où

$$f(x) = \int_a^x \frac{(x-z)^{n-1}}{(n-1)!}\varphi(z)\,dz.$$

En dérivant par rapport à x on trouvera

$$(26)\quad \frac{dG_n(x)}{dx} = \sum_{\nu=0}^{m} \frac{C_\nu^{(n)}}{2^\nu\,\nu!}\varphi^{(\nu+1)}(x) + \int_0^\infty \frac{\overline{E}_m^{(n)}(-t)}{m!}\varphi^{(m+2)}(x+t)\,dt$$

$$(27)\quad \frac{dF_n(x)}{dx} = \sum_{\nu=0}^{m+n} \frac{B_\nu^{(n)}}{\nu!} f^{(\nu+1)}(x) + \int_0^\infty \frac{\overline{B}_{m+n}^{(n)}(-t)}{(m+n)!}\varphi^{(m+2)}(x+t)\,dt$$

parce que les deux dernières intégrales convergent uniformément par rapport à x. D'autre part on aura en vertu de l'équation (25)

$$\int_a^x \varphi'(z) \overset{n}{\underset{\omega_1\cdots\omega_n}{\triangle}} z = \sum_{\nu=0}^{m+n} \frac{B_\nu^{(n)}}{\nu!}\psi^{(\nu)}(x) + \int_0^\infty \frac{\overline{B}_{m+n}^{(n)}(-t)}{(m+n)!}\varphi^{(m+2)}(x+t)\,dt,$$

où l'on a posé

$$\psi(x) = \int_a^x \frac{(x-z)^{n-1}}{(n-1)!}\varphi'(z)\,dz.$$

En intégrant par partie on trouve, si $\nu < n$

$$\psi^{(\nu-1)}(x) = \int_a^x \frac{(x-z)^{n-\nu}}{(n-\nu)!}\varphi'(z)\,dz = -\frac{(x-a)^{n-\nu}}{(n-\nu)!}\varphi(a) + \int_a^x \frac{(x-z)^{n-\nu-1}}{(n-\nu-1)!}\varphi(z)\,dz$$

$$= -\frac{(x-a)^{n-\nu}}{(n-\nu)!}\varphi(a) + f^{(\nu)}(x).$$

De plus on aura, si $\nu \geqq n$

$$\psi^{(\nu)}(x) = f^{(\nu+1)}(x).$$

Par conséquent

$$\sum_{\nu=0}^{m+n} \frac{B_\nu^{(n)}}{\nu!}\,[f^{(\nu+1)}(x)-\psi^{(\nu)}(x)] = \varphi(a)\sum_{\nu=0}^{n-1}\frac{B_\nu^{(n)}}{\nu!}\,\frac{(x-a)^{n-\nu-1}}{(n-\nu-1)!}$$

$$= \varphi(a)\,\frac{B_{n-1}^{(n)}(x-a)}{(n-1)!}.$$

Les équations (26) et (27) peuvent donc s'écrire comme il suit

$$\frac{d}{dx}\int \varphi(x) \overset{n}{\underset{\omega_1\cdots\omega_n}{\nabla}} x = \int \varphi'(x) \overset{n}{\underset{\omega_1\cdots\omega_n}{\nabla}} x, \tag{28}$$

$$\frac{d}{dx}\int_a^x \varphi(z) \overset{n}{\underset{\omega_1\cdots\omega_n}{\triangle}} z = \int_a^x \varphi'(z) \overset{n}{\underset{\omega_1\cdots\omega_n}{\triangle}} z + \varphi(a)\,\frac{B_{n-1}^{(n)}(x-a)}{(n-1)!}. \tag{29}$$

En dérivant de nouveau par rapport à x on trouvera

$$\frac{d^2}{dx^2}\int_a^x \varphi(z) \overset{n}{\underset{\omega_1\cdots\omega_n}{\triangle}} z = \frac{d}{dx}\int_a^x \varphi'(z) \overset{n}{\underset{\omega_1\cdots\omega_n}{\triangle}} z + \varphi(a)\,\frac{B_{n-2}^{(n)}(x-a)}{(n-2)!}$$

$$= \int_a^x \varphi''(z) \overset{n}{\underset{\omega_1\cdots\omega_n}{\triangle}} z + \varphi'(a)\,\frac{B_{n-1}^{(n)}(x-a)}{(n-1)!} + \varphi(a)\,\frac{B_{n-2}^{(n)}(x-a)}{(n-2)!}.$$

En dérivant $m+n$ fois on obtient

$$\frac{d^{m+n}F_n(x)}{dx^{m+n}} = \int_a^x \varphi^{(m+n)}(z) \overset{n}{\underset{\omega_1\cdots\omega_n}{\triangle}} z + \sum_{\nu=0}^{n-1} \varphi^{(m+\nu)}(a)\,\frac{B_\nu^{(n)}(x-a)}{\nu!}. \tag{30}$$

Mais le second membre peut se réduire. On a en effet

$$\int_a^x \varphi^{(m+n)}(z) \overset{n}{\underset{\omega_1\cdots\omega_n}{\triangle}} z = (-1)^{\overset{\cdot\cdot}{n}}\,\omega_1\,\omega_2\cdots\omega_n \sum \varphi^{(m+n)}(x+\Omega)$$

$$+ \int_a^\infty \frac{B_{n-1}^{(n)}(x-z)}{(n-1)!}\,\varphi^{(m+n)}(z)\,dz.$$

En intégrant par partie on trouve, en tenant compte de l'équation (23),

$$\int_a^\infty \frac{B_{n-1}^{(n)}(x-z)}{(n-1)!}\,\varphi^{(m+n)}(z)\,dz = \lim_{x\to\infty}\varphi^{(m)}(x) - \sum_{\nu=0}^{n-1}\varphi^{(m+\nu)}(a)\,\frac{B_\nu^{(n)}(x-a)}{\nu!}.$$

L'équation (30) peut donc s'écrire comme il suit

$$\frac{d^{m+n}F_n(x)}{dx^{m+n}} = \lim_{x\to\infty}\varphi^{(m)}(x) + (-1)^n\,\omega_1\,\omega_2\cdots\omega_n\sum\varphi^{(m+n)}(x+\Omega).$$

La série au second membre tend évidemment vers zéro quand x augmente indéfiniment et $\varphi^{(m)}(x)$ tend vers une limite finie. De l'équation (28) on déduit de même que

$$\frac{d^m G_n(x)}{dx^m} = \sum\varphi^{(m)}(x) \underset{\omega_1\cdots\omega_n}{\overset{n}{\nabla}} x = \varphi^{(m)}(x) + \int_0^\infty \bar{E}_0^{(n)}(-z)\,\varphi^{(m+1)}(x+z)\,dz.$$

L'intégrale au dernier membre tend vers zéro quand $x\to\infty$. On voit donc que les fonctions $F_n(x)$ et $G_n(x)$ admettent, pour $x \geqq b$, des dérivées continues de tous les ordres. En particulier les dérivées

$$F_n^{(m+n)}(x) \text{ et } G_n^{(m)}(x)$$

tendent vers des limites finies quand x augmente indéfiniment et les dérivées d'ordre superieur à celle que nous venons d'indiquer tendent vers zéro quand $x\to\infty$. Supposons maintenant que, quel que soit p, on sait trouver un entier positif m tel que

$$\lim_{x\to\infty} x^p\varphi^{(\nu)}(x) = 0 \qquad \text{si } \nu > m.$$

Dans ces conditions il résulte de notre analyse qu'on a, quelque grand que soit p,

$$\lim_{x\to\infty} x^p G_n^{(\nu)}(x) = 0,$$

$$\lim_{x\to\infty} x^p F_n^{(\nu)}(x) = 0,$$

pourvu que ν ait été choisi suffisamment grand. Par conséquent, si l'on applique, un nombre quelconque de fois, aux fonctions $G_n(x)$ et $F_n(x)$ nos deux opérations de sommation on arrive toujours à des expressions convergentes.

27. Nous avons vu que

$$\overset{n}{\underset{\omega_1 \cdots \omega_n}{\nabla}} \sum \varphi(x) \overset{n}{\underset{\omega_1 \cdots \omega_n}{\nabla}} x = \varphi(x).$$

Considérons maintenant la somme de première espèce

$$\sum \left(\overset{n}{\underset{\omega_1 \cdots \omega_n}{\nabla}} \varphi(x) \right) \overset{n}{\underset{\omega_1 \cdots \omega_n}{\nabla}} x.$$

Cette somme est égale à

$$2^{-n} \sum_{s_n=0}^{s_n=1} \cdots \sum_{s_1=0}^{s_1=1} \sum \varphi(x + s_1 \omega_1 + \cdots + s_n \omega_n) \overset{n}{\underset{\omega_1 \cdots \omega_n}{\nabla}} x = \overset{n}{\underset{\omega_1 \cdots \omega_n}{\nabla}} \sum \varphi(x) \overset{n}{\underset{\omega_1 \cdots \omega_n}{\nabla}} x.$$

Mais la dernière expression est égale à la fonction $\varphi(x)$.

On aura donc

$$\sum \overset{n}{\underset{\omega_1 \cdots \omega_n}{\nabla}} \varphi(x) \overset{n}{\underset{\omega_1 \cdots \omega_n}{\nabla}} x = \varphi(x).$$

Posons $\varphi(x) = G_{n+p}(x | \omega_1, \omega_2, \ldots, \omega_{n+p})$. On aura

$$\overset{n}{\underset{\omega_{p+1} \cdots \omega_{p+n}}{\nabla}} \varphi(x) = G_p(x | \omega_1, \omega_2, \ldots, \omega_p).$$

Par conséquent

$$\sum G_p(x | \omega_1, \omega_2, \ldots, \omega_p) \overset{n}{\underset{\omega_{p+1} \cdots \omega_{p+n}}{\nabla}} x = G_{n+p}(x | \omega_1, \omega_2, \ldots, \omega_{n+p}).$$

Cette équation peut aussi s'écrire

$$\sum \left(\sum \varphi(x) \overset{p}{\underset{\omega_1 \cdots \omega_p}{\nabla}} x \right) \overset{n}{\underset{\omega_{p+1} \cdots \omega_{p+n}}{\nabla}} x = \sum \varphi(x) \overset{p+n}{\underset{\omega_1 \cdots \omega_{p+n}}{\nabla}} x.$$

Considérons de même la somme de seconde espèce

$$\sum_a^x \left(\overset{n}{\underset{\omega_1 \cdots \omega_n}{\Delta}} \varphi(z) \right) \overset{n}{\underset{\omega_1 \cdots \omega_n}{\Delta}} z.$$

Cette somme est égale à

$$\frac{(-1)^n}{\omega_1\,\omega_2\cdots\omega_n}\sum_{s_n=0}^{s_n=1}\cdots\sum_{s_1=0}^{s_1=1}(-1)^{s_1+\cdots+s_n}\int_a^x\varphi(z+\Omega)\overset{n}{\underset{\omega_1\cdots\omega_n}{\triangle}}z$$

$$=\frac{(-1)^n}{\omega_1\,\omega_2\cdots\omega_n}\sum_{s_n=0}^{s_n=1}\cdots\sum_{s_1=0}^{s_1=1}(-1)^{s_1+\cdots+s_n}\left[\int_a^{x+\Omega}\varphi(z)\overset{n}{\underset{\omega_1\cdots\omega_n}{\triangle}}z\right.$$

$$\left.-\int_a^{a+\Omega}\frac{B_{n-1}^{(n)}(x+\Omega-z)}{(n-1)!}\varphi(z)\,dz\right]$$

$$=\overset{n}{\underset{\omega_1\cdots\omega_n}{\triangle}}\int_a^x\varphi(z)\overset{n}{\underset{\omega_1\cdots\omega_n}{\triangle}}z$$

$$-\frac{(-1)^n}{\omega_1\cdots\omega_n}\sum_{s_n=0}^{s_n=1}\cdots\sum_{s_1=0}^{s_1=1}(-1)^{s_1+\cdots+s_n}\int_0^{\Omega}\frac{B_{n-1}^{(n)}(x-a+\Omega-z)}{(n-1)!}\varphi(a+z)\,dz.$$

Le premier terme dans le dernier membre est égal à $\varphi(x)$. Considérons le second terme. On aura

$$\int_0^{\Omega}\frac{B_{n-1}^{(n)}(x-a+\Omega-z)}{(n-1)!}\varphi(a+z)\,dz$$

$$=\sum_{\nu=0}^{n-1}\frac{B_\nu^{(n)}(x-a)}{\nu!}\int_0^{\Omega}\frac{(\Omega-z)^{n-1-\nu}}{(n-1-\nu)\nu!}\varphi(a+z)\,dz.$$

Soit $f(x)$ une fonction telle que

$$f^{(n)}(x)=\varphi(x).$$

En intégrant par partie on trouvera

$$\int_0^{\Omega}\frac{(\Omega-z)^{n-1-\nu}}{(n-1-\nu)!}\varphi(a+z)\,dz=f^{(\nu)}(a+\Omega)-\sum_{s=0}^{n-1-\nu}\frac{\Omega^s}{s!}f^{(\nu+s)}(a).$$

Mais la difference d'ordre n de la fonction x^s est nulle, si $s<n$, c'est-à-dire que

$$\sum_{s_n=0}^{s_n=1}\cdots\sum_{s_1=0}^{s_1=1}(-1)^{s_1+s_2+\cdots+s_n}\mathfrak{Q}^s=0,\qquad s<n.$$

En substituant les expressions précédentes on trouvera donc

$$\int_a^x \overset{n}{\underset{\omega_1\cdots\omega_n}{\triangle}}\varphi(z)\overset{n}{\underset{\omega_1\cdots\omega_n}{\triangle}}z=\varphi(x)-\sum_{\nu=0}^{n-1}\frac{B_\nu^{(n)}(x-a)}{\nu!}\overset{n}{\underset{\omega_1\cdots\omega_n}{\triangle}}f^{(\nu)}(a).$$

Mais puisque

$$\overset{n}{\underset{\omega_1\cdots\omega_n}{\triangle}}f(a)=\int_0^1 dt_n\cdots\int_0^1 dt_2\int_0^1\varphi(a+\omega_1t_1+\omega_2t_2+\cdots+\omega_nt_n)\,dt_1,$$

cette équation peut aussi s'écrire comme il suit

$$\int_a^x\left(\overset{n}{\underset{\omega_1\cdots\omega_n}{\triangle}}\varphi(z)\right)\overset{n}{\underset{\omega_1\cdots\omega_n}{\triangle}}z$$

$$=\varphi(x)-\sum_{\nu=0}^{n-1}\frac{B_\nu^{(n)}(x-a)}{\nu!}\int_0^1 dt_n\cdots\int_0^1\varphi^{(\nu)}(a+\omega_1t_1+\cdots+\omega_nt_n)\,dt_1.$$

Posons $\varphi(x)=F_n(x|\omega_1,\omega_2,\cdots,\omega_n)$. On aura

$$\overset{p}{\underset{\omega_1\cdots\omega_p}{\triangle}}F_n(x|\omega_1,\cdots,\omega_n)=F_{n-p}(x|\omega_{p+1},\ldots,\omega_n).$$

Par conséquent

$$\int_a^x F_{n-p}(z|\omega_{p+1},\cdots,\omega_n)\overset{p}{\underset{\omega_1\cdots\omega_p}{\triangle}}z=F_n(x|\omega_1,\omega_2,\cdots,\omega_n)$$

$$-\sum_{\nu=0}^{p-1}\frac{B_\nu^{(p)}(x-a|\omega_1,\ldots,\omega_p)}{\nu!}\int_0^1 dt_p\cdots\int_0^1 F_n^{(\nu)}(a+\omega_1t_1+\cdots+\omega_pt_p)\,dt_1.$$

En posant en particulier $p = 1$ on trouvera

$$\sum_a^x F_{n-1}(z|\omega_1, \ldots, \omega_{n-1}) \underset{\omega_n}{\triangle} z = F_n(x|\omega_1, \ldots, \omega_n) - \frac{1}{\omega_n} \int_a^{a+\omega_n} F_n(x|\omega_1, \ldots, \omega_n)\, dx.$$

28. Considérons en second lieu la somme

$$\sum_a^x \left(\underset{\omega_1 \ldots \omega_n}{\overset{n}{\triangle}} \varphi(z) \right) \underset{2\omega_1 \ldots 2\omega_n}{\overset{n}{\triangle}} z.$$

En raisonnant comme nous venons de le faire on démontre que cette somme est égale à

$$\underset{\omega_1 \ldots \omega_n}{\overset{n}{\triangle}} \sum_a^x \varphi(z) \underset{2\omega_1 \ldots 2\omega_n}{\overset{n}{\triangle}} z$$

$$- \sum_{\nu=0}^{n-1} \frac{B_\nu^{(n)}(x-a|2\omega_1, \ldots, 2\omega_n)}{\nu!} \int_0^1 dt_n \cdots \int_0^1 \varphi^{(\nu)}(a + \omega_1 t_1 + \cdots + \omega_n t_n)\, dt_1.$$

En tenant compte de l'equation (7 ter):

$$\text{(7 ter)} \qquad \underset{\omega_1 \ldots \omega_n}{\overset{n}{\triangle}} \sum_a^x \varphi(z) \underset{2\omega_1 \ldots 2\omega_n}{\overset{n}{\triangle}} z = \sum \varphi(x) \underset{\omega_1 \ldots \omega_n}{\overset{n}{\nabla}} x,$$

on trouvera donc la relation suivante

$$\text{(31)} \qquad \sum_a^x \left(\underset{\omega_1 \ldots \omega_n}{\overset{n}{\triangle}} \varphi(z) \right) \underset{2\omega_1 \ldots 2\omega_n}{\overset{n}{\triangle}} z = \sum \varphi(x) \underset{\omega_1 \ldots \omega_n}{\overset{n}{\nabla}} x$$

$$- \sum_{\nu=0}^{n-1} \frac{B_\nu^{(n)}(x-a|2\omega_1, \ldots, 2\omega_n)}{\nu!} \int_0^1 dt_n \cdots \int_0^1 \varphi^{(\nu)}(a + \omega_1 t_1 + \cdots + \omega_n t_n)\, dt_1.$$

Remplaçons dans cette équation la fonction $\varphi(x)$ par la fonction $F_n(x|\omega_1, \omega_2, \ldots, \omega_n)$ et rappelons la relation (15):

$$(15)\quad \int_0^1 dt_n \cdots \int_0^1 F_n(x+\omega_1 t_1+\cdots+\omega_n t_n)\,dt_1 = \int_a^x \frac{(x-z)^{n-1}}{(n-1)!}\varphi(z)\,dz.$$

Cette intégrale est une fonction de x qui s'annule dans le point $x=a$ et il en est de même de ses dérivées des $(n-1)$ premiers ordres. Le dernier terme dans le second membre de l'équation (31) disparaît donc et cette équation prend la forme

$$(32)\quad \sum F_n(x\,|\,\omega_1, \omega_2, \ldots, \omega_n) \underset{\omega_1\ldots\omega_n}{\overset{n}{\nabla}} x = F_n(x\,|\,2\omega_1, 2\omega_2, \ldots, 2\omega_n),$$

ou ce qui revient au même

$$\sum \left(\sum_a^x \varphi(x) \underset{\omega_1\ldots\omega_n}{\overset{n}{\triangle}} x \right) \underset{\omega_1\ldots\omega_n}{\overset{n}{\nabla}} x = \sum_a^x \varphi(x) \underset{2\omega_1\ldots 2\omega_n}{\overset{n}{\triangle}} x.$$

Soit p un entier $\leqq n$. Par un raisonnement semblable on démontre la relation plus générale

$$\sum F_n(x\,|\,\omega_1, \omega_2, \ldots, \omega_n) \underset{\omega_1\ldots\omega_p}{\overset{p}{\nabla}} x = F_n(x\,|\,2\omega_1, 2\omega_2, \ldots, 2\omega_p, \omega_{p+1}, \ldots, \omega_n).$$

29. Considérons en dernier lieu la somme de seconde espèce

$$\sum_a^x \left(\underset{\omega_1\ldots\omega_n}{\overset{n}{\nabla}} \varphi(z) \right) \underset{2\omega_1\ldots 2\omega_n}{\overset{n}{\triangle}} z.$$

Cette somme est égale à

$$\underset{\omega_1\ldots\omega_n}{\overset{n}{\nabla}} \sum_a^x \varphi(z) \underset{2\omega_1\ldots 2\omega_n}{\overset{n}{\triangle}} z$$

$$+ 2^{-n} \sum_{s_n=0}^{s_n=1} \cdots \sum_{s_1=0}^{s_1=1} \int_0^{\Omega} \frac{B_{n-1}^{(n)}(x-a+\Omega-z\,|\,2\omega_1, \ldots, 2\omega_n)}{(n-1)!}\,\varphi(a+z)\,dz.$$

En tenant compte de l'équation (6 ter):

$$\text{(6 ter)} \qquad \overset{n}{\underset{\omega_1 \dots \omega_n}{\nabla}} \int_a^x \varphi(z) \overset{n}{\underset{2\omega_1 \dots 2\omega_n}{\Delta}} z = \int_a^x \varphi(z) \overset{n}{\underset{\omega_1 \dots \omega_n}{\Delta}} z,$$

et en posant

$$f(x) = \int_a^x \frac{(x-z)^{n-1}}{(n-1)!} \varphi(z)\, dz,$$

on trouvera donc la relation suivante

$$\int_a^x \left(\overset{n}{\underset{\omega_1 \dots \omega_n}{\nabla}} \varphi(z) \right) \overset{n}{\underset{2\omega_1 \dots 2\omega_n}{\Delta}} z = \int_a^x \varphi(z) \overset{n}{\underset{\omega_1 \dots \omega_n}{\Delta}} z$$

$$- \sum_{\nu=0}^{n-1} \frac{B_\nu^{(n)}(x-a \mid 2\omega_1, \dots, 2\omega_n)}{\nu!} \overset{n}{\underset{\omega_1 \dots \omega_n}{\nabla}} f^{(\nu)}(a).$$

Remplaçons dans cette équation la fonction $\varphi(x)$ par la fonction $G_n(x \mid \omega_1, \omega_2, \dots, \omega_n)$; on trouve, après quelques réductions, la relation suivante

$$F_n(x \mid 2\omega_1, 2\omega_2, \dots, 2\omega_n) = \int_a^x G_n(z \mid \omega_1, \omega_2, \dots, \omega_n) \overset{n}{\underset{\omega_1 \dots \omega_n}{\Delta}} z$$

$$+ \sum_{\nu=0}^{n-1} \frac{B_\nu^{(n)}(x-a \mid \omega_1, \omega_2, \dots, \omega_n)}{\nu!} \xi^{(\nu)}(a),$$

où l'on a posé pour abréger

$$\xi(x) = \int_0^1 dt_n \cdots \int_0^1 dt_2 \int_0^1 F_n(x + \omega_1 t_1 + \omega_2 t_2 + \cdots + \omega_n t_n \mid 2\omega_1, 2\omega_2, \dots, 2\omega_n)\, dt_1.$$

Et la valeur de cette intégrale se détermine par l'équation

$$\int_0^1 dt_n \cdots \int_0^1 F_n(x+\omega_1 t_1+\cdots+\omega_n t_n \mid 2\omega_1,\ldots,2\omega_n)\,dt_1 = \mathfrak{S}\left(\int_a^x \frac{(x-z)^{n-1}}{(n-1)!}\varphi(z)\,dz\right)_{\omega_1\ldots\omega_n}^{n\,\nabla} x.$$

Polynomes de Bernoulli et polynomes d'Euler

30. Considérons à titre d'exemple le cas particulier où la fonction $\varphi(x)$ est égale à x^ν, ν étant un entier non négatif. Les développements (24) et (25) se réduisent à un nombre fini de termes car en prenant $m=\nu$ le terme complémentaire disparaît. La somme de première espèce est donc égale à un polynome d'Euler et la somme de seconde espèce s'exprime par un polynome de Bernoulli. On peut vérifier ce résultat directement de la manière suivante. On aura pour toute valeur positive de η

$$\sum(-1)^{s_1+s_2+\cdots+s_n}e^{-\eta(x+\varphi)} = \frac{e^{-\eta x}}{(1+e^{-\eta\omega_1})(1+e^{-\eta\omega_2})\cdots(1+e^{-\eta\omega_n})},$$

$$\sum e^{-\eta(x+\varphi)} = \frac{e^{-\eta x}}{(1-e^{-\eta\omega_1})(1-e^{-\eta\omega_2})\cdots(1-e^{-\eta\omega_n})},$$

les sommations étant étendues sur toutes les valeurs entières non négatives de $s_1, s_2, \ldots, s_n$. En dérivant ν fois par rapport à η on trouvera

(33) $$\sum(-1)^{s_1+s_2+\cdots+s_n}(x+\Omega)^\nu e^{-\eta(x+\varphi)} = (-1)^\nu D_\eta^\nu \frac{e^{-\eta x}}{(1+e^{-\eta\omega_1})(1+e^{-\eta\omega_2})\cdots(1+e^{-\eta\omega_n})},$$

(34) $$\sum(x+\Omega)^\nu e^{-\eta(x+\varphi)} = (-1)^\nu D_\eta^\nu \frac{e^{-\eta x}}{(1-e^{-\eta\omega_1})(1-e^{-\eta\omega_2})\cdots(1-e^{-\eta\omega_n})}.$$

La première série est donc une fonction de η qui est holomorphe dans le point $\eta=0$, mais la seconde série est une fonction de η qui admet le point $\eta=0$ comme pôle d'ordre $n+\nu$. Dans les seconds membres de ces deux équations

figurent les fonctions génératrices des polynomes d'Euler et des polynomes de Bernoulli. En développant suivant les puissances de η dans le second membre de l'équation (33) on obtient

$$G_n(x \mid \omega_1, \omega_2, \ldots, \omega_n; \eta) = (-1)^\nu D_\eta^\nu \sum_{s=0}^{\infty} (-1)^s \frac{\eta^s}{s!} E_s^{(n)}(x),$$

$$= E_\nu^{(n)}(x) - \frac{\eta}{1} E_{\nu+1}^{(n)}(x) + \cdots.$$

Faisons tendre η vers zéro, il vient

$$\sum_{\omega_1 \ldots \omega_n} x^\nu \overset{n}{\nabla} x = E_\nu^{(n)}(x \mid \omega_1, \omega_2, \ldots, \omega_n).$$

Pour trouver la somme de seconde espèce il nous reste à déterminer la valeur de l'intégrale qui figure dans l'expression de $F_n(x; \eta)$:

$$F_n(x; \eta) = \int_0^\infty \frac{B_{n-1}^{(n)}(x-z)}{(n-1)!} z^\nu e^{-\eta z}\, dz + (-1)^n \omega_1 \ldots \omega_n \sum (x+\Omega)^\nu e^{-\eta(x+\Omega)}.$$

On aura pour toute valeur positive de η

$$\int_0^\infty z^\nu e^{-\eta z}\, dz = \frac{\nu!}{\eta^{\nu+1}}.$$

Et le polynome de Bernoulli se développe de la manière suivante:

$$B_{n-1}^{(n)}(x-z) = B_{n-1}^{(n)}(x) - \binom{n-1}{1} z\, B_{n-2}^{(n)}(x)$$

$$+ \binom{n-1}{2} z^2 B_{n-3}^{(n)}(x) - \cdots + (-1)^{n-1} z^{n-1} B_0^{(n)}(x).$$

Par conséquent

$$\int_0^\infty \frac{B_{n-1}^{(n)}(x-z)}{(n-1)!} e^{-\eta z}\, dz = (-1)^{n-1} \sum_{s=0}^{n-1} (-1)^s \frac{B_s^{(n)}(x)}{s!} \eta^{s-n}.$$

En dérivant ν fois par rapport à η on trouvera

$$\int_0^\infty \frac{B_{n-1}^{(n)}(x-z)}{(n-1)!} z^\nu e^{-\eta z}\, dz = -(-1)^{\nu+n} D_\eta^\nu \sum_{s=0}^{n-1} (-1)^s \frac{B_s^{(n)}(x)}{s!} \eta^{s-n}.$$

De même, en développant le second membre de l'équation (34) suivant les puissances de η on obtient

$$\omega_1 \omega_2 \cdots \omega_n \sum (x+\Omega)^\nu e^{-\eta(x+\Omega)} = (-1)^\nu D_\eta^\nu \sum_{s=0}^{\infty} (-1)^s \frac{B_s^{(n)}(x)}{s!} \eta^{s-n}.$$

En substituant ces développements dans l'expression de $F_n(x;\eta)$ on trouvera

$$F_n(x;\eta) = (-1)^{\nu+n} D_\eta^\nu \sum_{s=n}^{\infty} (-1)^s \frac{B_s^{(n)}(x)}{s!} \eta^{s-n}.$$

La fonction $F_n(x;\eta)$ est donc holomorphe dans le point $\eta = 0$. Faisons tendre η vers zéro, il vient

$$\sum_0^x x^\nu \underset{\omega_1 \ldots \omega_n}{\overset{n}{\triangle}} x = \frac{B_{\nu+n}^{(n)}(x \mid \omega_1, \omega_2, \ldots, \omega_n)}{(\nu+1)(\nu+2)\cdots(\nu+n)}.$$

En particulier, pour $\nu = 0$, on trouvera

$$\sum_a^x \underset{\omega_1 \ldots \omega_n}{\overset{n}{\triangle}} x = \frac{B_n^{(n)}(x-a \mid \omega_1, \omega_2, \ldots, \omega_n)}{n!}.$$

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DIFFERENTIAL GEOMETRY OF AN m-DIMENSIONAL MANIFOLD IN A EUCLIDEAN SPACE OF n DIMENSIONS*

BY

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1. Introduction

This paper had its inspiration in the Einstein Theory. It grew out of an attempt to give geometric interpretation to the Einstein equations for material free space. It was first developed by the methods of Voss†, and Beez‡, but later changed to the simpler notation of Wilson and Moore§, which is a combination of vector analysis with the absolute calculus of Ricci and Levi-Civita||. In Voss is to be found the differential geometry of varieties of $n-1$ dimensions in a space of n dimensions. In the article of Wilson and Moore is developed the geometry of a two-dimensional manifold in n dimensions. These are the two natural extensions of surface theory to a higher number of dimensions. The object of this paper is to begin the study of the intermediate cases.

The method of the absolute calculus is developed in great detail in Ricci's *Lezioni sulla Teoria delle Superficie,* and is there applied to the ordinary surface theory. A sufficient introduction for the purposes of this paper will be found in the paper of Wilson and Moore above mentioned. It is their method that is followed in the deduction of the integrability conditions. It was apparently first given by Ricci in the *Lezioni,* pp. 87—91.

We shall denote by V_m a variety of m dimensions, defined in a euclidean space P_n of n dimensions by means of $n-m$ equations

$$F_\alpha(x_1, x_2, \cdots, x_n) = 0 \qquad (\alpha = 1, 2, 3, \cdots, n-1).$$

* Presented to the Society, December 28, 1921.

† A. Voss, *Zur Theorie der Transformation quadratischer Differentialausdrücke und der Krümmung höherer Mannigfaltigkeiten,* Mathematische Annalen, vol. 16 (1880).

‡ Beez, Zeitschrift für Mathematik und Physik, vol. 24 (1879), pp. 1, 65.

§ E. B. Wilson and C. L. E. Moore, *Differential geometry of two-dimensional surfaces in hyperspace,* Proceedings of the American Academy of Arts and Sciences, vol. 52 (1916), p. 269.

|| M. M. G. Ricci and T. Levi-Civita, *Méthodes de calcul différential absolu et leurs applications,* Mathematische Annalen, vol. 54 (1900—1901), pp. 124—201. This paper will be referred to in what follows by the term *Absolute Calculus.*

Let u_r be a set of variables such that

$$x_i = x_i(u_1, u_2, \cdots, u_m) \qquad (i = 1, 2, 3, \cdots, n)$$

make the F's identically zero in the u's. We have then for the differential of arc in V_m

$$ds^2 = \sum_{rs} a_{rs} du_r du_s \qquad (r, s = 1, 2, 3, \cdots, m) \tag{1}$$

in which

$$a_{rs} = a_{sr} = \sum_i \frac{\partial x_i}{\partial u_r} \frac{\partial x_i}{\partial u_s} \qquad (i = 1, 2, 3, \cdots, n).$$

In our euclidean n space P_n let us denote by i_h the unit vector parallel to the x_h axis. A variable point in P_n is given by the vector x having components x_h parallel to the coördinate axes, or

$$x = \sum_h x_h i_h.$$

We shall use subscripts to distinguish between vectors and also to denote covariant differentiation with regard to the fundamental quadratic differential form (1), but when it becomes necessary to use subscripts in both senses on the same letter those used for the latter purpose will always follow those used for the former, and the two kinds of subscripts will be separated by a comma.*

2. Deduction of integrability conditions

We have

$$a_{rs} = x_r \cdot x_s.$$

Differentiating this covariantly we obtain

$$x_{rt} \cdot x_s + x_r \cdot x_{st} = 0,$$

for the first covariant derivative of the fundamental form is zero. Also the second covariant derivative of a scalar is symmetric so that

$$x_{rs} = x_{sr},$$

and from these two sets of equations is easily deduced

$$x_r \cdot x_{st} = 0,$$

which is to say that x_{st} is normal to V_m.

* Although using Clarendon type for vectors seems preferable to the author, no ambiguity arises in the present paper from not doing so.

The normal to the variety F_α is ∇F_α and we shall denote by ξ_α the unit vector in this direction. The $n-m$ vectors ξ_α define the euclidean $n-m$ space, P_{n-m}, normal to V_m. The subspace of this defined by the x_{rs} will be called the *principal normal space.* It is convenient to make use also of another set of vectors ζ_α that may be used to define this P_{n-m}. We shall call them the polar set, and they shall be defined by the following equations, from which their geometric relation to the vectors ξ_α is at once evident:

$$\zeta_\alpha \cdot \xi_\beta = 0, \quad \alpha \neq \beta, \qquad \zeta_\alpha \cdot \xi_\alpha = 1. \tag{2}$$

We have also the relations

$$\xi_\alpha \cdot x_r = 0, \qquad \zeta_\alpha \cdot x_r = 0.$$

Whence by covariant differentiation is obtained

$$\xi_{\alpha,r} \cdot x_s + \xi_\alpha \cdot x_{rs} = 0, \tag{3}$$

$$\zeta_{\alpha,r} \cdot x_s + \zeta_\alpha \cdot x_{rs} = 0, \tag{4}$$

$$\xi_{\alpha,r} \cdot \zeta_\beta + \xi_\alpha \cdot \zeta_{\beta,r} = 0. \tag{5}$$

Since the x_{rs} are normal to V_m they can be expressed in terms of the ξ_α or the ζ_α. It seems more convenient to use the latter. Hence let

$$x_{rs} = \sum_\alpha b_{\alpha rs} \zeta_\alpha. \tag{6}$$

If this be multiplied by ξ_α the following is obtained:

$$\xi_\alpha \cdot x_{rs} = b_{\alpha rs} = -\xi_{\alpha,r} \cdot x_s,$$

in which the latter equality is obtained by using (3). In (5) set

$$\xi_\alpha \cdot \zeta_{\beta,r} = \mu_{\alpha\beta r} = -\xi_{\alpha,r} \cdot \zeta_\beta.$$

These two sets of equations may be solved for $\xi_{\alpha,r}$ getting

$$\xi_{\alpha,r} = -\sum_t b_{\alpha rt} x^t - \sum_\beta \mu_{\alpha\beta r} \xi_\beta. \tag{7}$$

Now differentiate (6) covariantly,

$$x_{rst} = \sum_a b_{ars,t}\zeta_a + \sum_a b_{ars}\zeta_{a,t}.$$

The vectors ζ can of course be expressed in terms of the ξ, and hence let

$$\zeta_a = \sum_\gamma \nu_{a\gamma}\xi_\gamma, \tag{8}$$

in which $\nu_{a\gamma}$ is symmetric. Now differentiate (8) covariantly, getting

$$\zeta_{a,t} = \sum_\gamma \nu_{a\gamma,t}\xi_\gamma + \sum_\gamma \nu_{a\gamma}\xi_{\gamma,t},$$

and finally place these values in the above expression for x_{rst} and simplify by means of (7). Thus is obtained the equation

$$x_{rst} = \sum_\gamma \Big(\sum_a b_{ars,t}\nu_{a\gamma} + \sum_a b_{ars}\nu_{a\gamma,t} - \sum_{a\beta} b_{ars}\nu_{a\beta}\mu_{\beta\gamma t}\Big)\xi_\gamma - \sum_{a\gamma u} b_{ars}b_{\gamma tu}\nu_{a\gamma}x^u.$$

A theorem of the *Absolute Calculus* gives us the following relation between the third covariant derivatives of a function

$$x_{rst} - x_{rts} = \sum_u x^u(ur,st)$$

in which (ur,st) is the Riemann symbol of the first kind. Hence when this difference is formed the resulting coefficients of ξ_γ must be zero and the coefficient of x^u must be (ur,st). Thus are obtained the integrability conditions

$$(ur,st) = \sum_{a\beta}\nu_{a\beta}(b_{art}b_{\beta su} - b_{ars}b_{\beta tu}), \tag{9}$$

$$\begin{aligned}\sum_a \nu_{a\gamma}[b_{ars,t} - b_{art,s}] &= \sum_{a\beta}\nu_{a\beta}(b_{ars}\mu_{\beta\gamma t} - b_{art}\mu_{\beta\gamma s}) \\ &\quad - \sum_a(\nu_{a\gamma,t}b_{ars} - \nu_{a\gamma,s}b_{art}).\end{aligned} \tag{10}$$

If the defining equations $F_a = 0$ are so chosen that the ∇F_a are mutually perpendicular vectors, as is done in substance by Ricci, these equations have the simpler form

$$(ur,st) = \sum_a(b_{art}b_{asu} - b_{ars}b_{atu}),$$

$$b_{ars,t} - b_{art,s} = \sum_a(b_{\beta rs}\mu_{\beta at} - b_{\beta rt}\mu_{\beta as}).$$

3. Orthogonal nets of curves in the variety

In order to discuss curvature we shall introduce an orthogonal m-ple, after the manner of the absolute calculus.

Let λ^r be m functions of the u's which do not vanish in the neighborhood in which we are interested. By means of the equations

$$\frac{du_1}{\lambda^1} = \frac{du_2}{\lambda^2} = \cdots = \frac{du_m}{\lambda^m}$$

they define a family of curves on the variety, one through each point. If all the λ's are multiplied by the same function of the u's it does not change the family of curves represented, and so in particular we can so choose our λ's that they satisfy the relation

$$\sum_{rs} a_{rs}\lambda^r\lambda^s = \sum_r \lambda_r\lambda^r = \sum_{rs} a^{rs}\lambda_r\lambda_s = 1. \tag{11}$$

When this is done the λ^r are called the contravariant coördinates of the family of curves. In this manner m such families may be defined, orthogonal each to each. Such a set of families of curves is called an orthogonal m-ple. Indicating the family by the first subscript we have the following relations between the covariant and contravariant coördinates of the families of an orthogonal m-ple:

$$\sum_{rs} a_{rs}\lambda_h^r\lambda_k^s = \sum_r \lambda_{hr}\lambda_k^r = \sum_{rs} a^{rs}\lambda_{hr}\lambda_{ks} = \begin{cases} 0 & k \neq h, \\ 1 & k = h. \end{cases} \tag{12}$$

The orthogonal m-ple has the following invariants,

$$I_{ahk} = \sum_{rs} b_{ars}\lambda_h^r\lambda_k^s,$$

by means of which the b_{ars} may be written in terms of the m-ple

$$b_{ars} = \sum_{hk} I_{ahk}\lambda_{hr}\lambda_{ks}.$$

Denote by b_a the quadratic differential form

$$b_a = \sum_{rs} b_{ars}\,du_r\,du_s.$$

These forms correspond to the second fundamental form in surface theory. They may be combined into a vector form

$$b = \sum_a b_a \zeta_a = \sum_{rs} x_{rs}\, du_r\, du_s, \tag{13}$$

which will be known as the second fundamental form of the variety V_m in what follows. In terms of the m-ple

$$b = \sum_{hkrs} I_{hk}\, \lambda_{hr}\, \lambda_{ks}\, du_r\, du_s, \tag{14}$$

in which

$$I_{hk} = \sum_a I_{ahk}\, \zeta_a,$$

we also find

$$x_{rs} = \sum_{hr} I_{hk}\, \lambda_{hr}\, \lambda_{ks}. \tag{15}$$

4. Curvature of the variety

The unit vectors tangent to the curves λ_h^r are

$$\eta_h = \sum_r \lambda_h^r\, x_r \tag{16}$$

and since the curves are those of an orthogonal m-ple these vectors are mutually perpendicular. The rate of change of these vectors may be found by covariant differentiation

$$\eta_{h,r} = \sum_s \lambda_h^s\, x_{rs} + \sum_s \lambda_{hs,r}\, x^s.$$

Now by *Absolute Calculus*, p. 149,

$$\lambda_{hr,s} = \sum_{is} \gamma_{hij}\, \lambda_{ir}\, \lambda_{js},$$

and if we substitute this and (15) in the above we get

$$\eta_{h,r} = \sum_i I_{ih}\, \lambda_{ir} + \sum_{ij} \gamma_{hij}\, \lambda_{jr}\, \eta_i.$$

Now the curvature of the curves λ_h^r is

$$K_h = \frac{d\eta_h}{ds_h} = \sum_r \eta_{h,r} \frac{du_r}{ds_h} = \sum_r \eta_{h,r}\, \lambda_h^r = I_{hh} + \sum_i \gamma_{hih}\, \eta_i \tag{17}$$

$$= I_{hh} - \sum_i \gamma_{ihh}\, \eta_i.$$

Also* $\gamma_{ihh} = 0$ is the condition that the family of curves λ_h^r be geodesics. Hence the curvature of a curve on the variety consists of two parts, a vector part I_{hh} normal to the variety, which is the same for all curves having the same direction and so may be considered as the normal curvature of the variety in that direction, and a vector part in the variety which measures the deviation from the geodesic in that direction, and so is the geodesic curvature of the curve.

By an argument similar to that used by Wilson and Moore† one easily proves the two theorems:

If a curve is projected on any P_k through the tangent line to the curve, the curvature of the projection at the point of tangency is equal to the projection of the curvature at that point.

The projection of the curvature of any section of the variety made by a P_{n-m+1} on the normal P_{n-m+1} passing through the same direction on the variety is equal to the curvature of the normal section.

If we compute the rate of change of any unit normal, in the variety, to our curve, we have

$$\frac{d\eta_k}{ds_h} = \sum_r \eta_{k,r}\lambda_h^r = I_{hk} - \sum_i \gamma_{ihk}\eta_i,$$

which gives the interpretation of the invariants I_{hk}.

If the invariant of the variety $\sum_{rs} a^{rs}x_{rs}$ be computed in terms of the invariants of the net it is found that

$$\sum_{rs} a^{rs}x_{rs} = \sum_{hk} a^{rs}I_{hk}\lambda_{hr}\lambda_{ks} = \sum_h I_{hh}.$$

In other words, the sum of the normal curvatures in any m mutually orthogonal directions in the variety is independent of the directions. It is then a numerical measure of the curvature of the variety, being a natural extension of the notion of mean curvature of a surface in three dimensions. We shall set

$$h = \frac{1}{m}\sum_h I_{hh} = \frac{1}{m}\sum_{rs} a^{rs}x_{rs}, \tag{18}$$

and shall call h the mean vector curvature of the variety.

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* *Absolute Calculus*, p. 154.

† Loc. cit., p. 320 and p. 321.

5. Curvature of the variety with respect to a given normal direction

The above vector h may be obtained in several different ways, some of which seem to help in forming a conception of the geometry of the situation. We shall take up first the notion of curvature with respect to a given normal direction. In ordinary surface theory the normal curvature of a surface may be found for any direction through the surface point by taking a neighboring point in that direction and finding its distance from the tangent plane. Calling this distance p, the curvature is

$$K = \frac{2p}{ds^2},$$

which may be put into vector notation

$$K = -\frac{dn \cdot dx}{ds^2} = -\frac{dn \cdot dx}{dx \cdot dx}.$$

Similarly, at any point of our variety we may choose a normal direction and define it by a unit vector σ. Through the variety point there is a unique P_{n-1} perpendicular to σ and it is of course tangent to the variety. If we take for our p in the above formula the perpendicular distance from a neighboring point in the variety to this P_{n-1}, we have defined curvature with respect to the direction σ. Written in terms of vectors this definition gives

$$K_\sigma = -\frac{d\sigma \cdot dx}{dx \cdot dx} = -\sum_{rs} \sigma_r \cdot x_s \lambda^r \lambda^s.$$

We now determine those directions in the variety for which K is stationary, which we shall call the directions of principal curvature of the variety with respect to the direction σ. Since σ is perpendicular to the variety, $\sigma \cdot x_r = 0$, and by covariant differentiation

$$\sigma_s \cdot x_r + \sigma \cdot x_{rs} = 0,$$

so that the curvature may be written in the form

$$(19) \qquad K = -\sum_{rs} \sigma_r \cdot x_s \lambda^r \lambda^s = \sum_{rs} \sigma \cdot x_{rs} \lambda^r \lambda^s = \frac{\sigma \cdot b}{dx \cdot dx}.$$

By differentiation of the second expression for K, remembering that the λ's are connected by the relation (11), we find that the required directions are given by

$$\sum_s (\sigma \cdot x_{rs} - \varrho a_{rs}) \lambda^s = 0, \tag{20}$$

in which ϱ is a root of the determinantal equation

$$|\sigma \cdot x_{rs} - \varrho a_{rs}| = 0. \tag{21}$$

From the form of these equations we know that these directions are mutually orthogonal, and so form an orthogonal m-ple which we shall call the m-ple of curvature with respect to the direction σ. Using this m-ple as m-ple of reference we have*

$$\sigma \cdot x_{rs} = \sum_h \varrho_h \lambda_{hr} \lambda_{hs},$$

where the ϱ's are the roots of (21). Now (15) gives x_{rs} in terms of any m-ple, so for this particular one

$$\sigma \cdot I_{hk} = 0, \quad k \neq h, \quad \varrho_h = \sigma \cdot I_{hh}. \tag{22}$$

Moreover the principal curvatures are precisely the ϱ_h, so that we have the theorem:

The curvature of a curve of the m-ple of curvature with respect to the direction σ is the projection of its vector normal curvature on the direction σ.

The mean of these principal curvatures is given by

$$\frac{1}{m} \sum_h \varrho_h = \frac{1}{m} \sum_{rs} a^{rs} \sigma \cdot x_{rs} = \sigma \cdot h.$$

For a particular direction σ, this mean curvature has a maximum, which may be found by differentiation and when so found turns out to be precisely the direction h. The m-ple of curvature with regard to this direction we call the principal m-ple of curvature of the variety.

From the above expression for the mean curvature it is obvious that the mean curvature of the variety with respect to any direction σ perpendicular to h is zero.

Many of the theorems of ordinary differential geometry may be extended by means of this notion of curvature with respect to a given normal direction σ, for example:

* *Absolute Calculus*, p. 159.

If the normals, σ, along any curve form a developable then the curve is a line of curvature with respect to the direction σ.

The proof is as follows

$$Y = x + t\sigma$$

is any point on such a normal. If we take a similar line at an infinitesimal distance along any curve λ^r we have

$$dY = dx + t\,d\sigma + \sigma\,dt.$$

If these lines are to form a developable there must be some value of t for which dY has the same direction as σ. For this value

$$dx + t\,d\sigma + \sigma\,dt = k\sigma.$$

And if we multiply by σ we find

$$dt = k,$$

whence

$$dx + t\,d\sigma = 0,$$

which gives

$$\sum_r (x_r + t\sigma_r)\lambda^r = 0$$

and multiplying by x_s we obtain

$$\sum_r (a_{rs} + t\sigma_r \cdot x_s)\lambda^r = 0.$$

But since $\sigma \cdot x_r = 0$ it follows that

$$\sigma_r \cdot x_s + \sigma \cdot x_{rs} = 0$$

and this further simplifies our equation to

$$\sum_r (a_{rs} - t\sigma \cdot x_{rs})\lambda^r = 0,$$

which is equivalent to (20) and so proves the theorem.

6. The asymptotic cone and the Dupin indicatrix

We may call the directions in which the tangent P_{n-1} perpendicular to σ intersects V_m the asymptotic directions with respect to the direction σ. These directions form a second degree cone of $m-1$ dimensions which we call the

asymptotic cone. Similarly, if this P_{n-1} is infinitesimally displaced but kept normal to σ it will intersect V_m in a quadratic manifold which we shall call the Dupin indicatrix with respect to the direction σ. When the direction σ is that of h we call the asymptotic cone the principal asymptotic cone and the indicatrix the principal Dupin indicatrix.

Analytically

$$(X-x)\cdot\sigma = 0$$

is the tangent P_{n-1}. Displaced slightly it becomes

$$(X-x)\cdot\sigma = \epsilon.$$

If V_m is expanded in the neighborhood of the point x we have

$$X = x+\sum_r \frac{\partial x}{\partial u_r}(u_r-u_r^0)+\tfrac{1}{2}\sum_{rs}\frac{\partial^2 x}{\partial u_r \partial u_s}(u_r-u_r^0)(u_s-u_s^0)+\cdots,$$

and we have for the intersection

$$\tfrac{1}{2}\Big[\sum_{rs}\frac{\partial^2 x}{\partial u_r \partial u_s}(u_r-u_r^0)(u_s-u_s^0)\Big]\cdot\sigma = \epsilon,$$

which simplifies to

$$\sum_{rs}\sigma\cdot x_{rs}\,v_r\,v_s = \epsilon$$

in which

$$v_r = u_r - u_r^0.$$

If we wish to replace the v_r by variables measured along an arbitrary orthogonal m-ple we can write the indicatrix as

$$\sum_{ij}\sigma\cdot I_{ij}\,w_i\,w_j = \epsilon.$$

The asymptotic cone with regard to the direction σ is thus defined by

$$\sigma\cdot b = 0,$$

and the principal asymptotic cone by

$$h\cdot b = 0.$$

We shall call a curve on V_m an asymptotic with respect to the direction σ if its tangent at every point belongs to the asymptotic cone with respect to the direction σ at that point.

If we introduce an orthogonal m-ple we find that the condition that the curve λ_m^r be an asymptotic is

$$\sigma \cdot I_{mm} = 0$$

whence we conclude:

A necessary and sufficient condition that a curve on V_m be an asymptotic with respect to the direction σ is that its curvature be normal to σ.

From which follows immediately:

An asymptotic with respect to the direction σ is characterized by the property that its osculating plane is normal to σ.

The discriminant of the Dupin indicatrix is

$$|\sigma \cdot x_{rs}| = 0.$$

Whence the lines of this cone are the directions in space with respect to which the asymptotic cone has multiple elements.

Another theorem seems worthy of note:

If an orthogonal m-ple can be picked from the principal asymptotic directions, then V_m is a minimal variety.

For taking this m-ple as m-ple of reference

$$\sum_{rs} h \cdot x_{rs} \lambda_i^r \lambda_i^s = 0,$$

or

$$h \cdot I_{ii} = 0$$

for every i. But h is a linear combination of the I_{ii} and so this relation cannot hold unless $h = 0$. On the other hand $h = 0$ is precisely the condition for a minimal variety.*

7. Conjugate directions

Let us denote by φ the dyadic

$$\varphi \equiv \sum_{ij} \sigma \cdot I_{ij} \eta_i \eta_j.$$

* C. L. E. Moore, *Note on minimal varieties in hyperspace*, Bulletin of the American Mathematical Society, vol. 27 (1921), p. 216.

Then if $r = \sum_i w_i \eta_i$, the indicatrix may be written

$$r \cdot \varphi \cdot r = \varepsilon.$$

Now let η be any given vector in the tangent P_m, and τ a variable vector also in it. The line $\tau + s\eta$ will intersect the indicatrix in points determined by

$$(\tau + s\eta) \cdot \varphi \cdot (\tau + s\eta) = \varepsilon.$$

If now τ is the vector to the mid point of the chord, then the coefficient of the first degree term in s must be zero, or

$$\tau \cdot \varphi \cdot \eta = 0$$

defines the locus of the mid points of the chords of φ that are parallel to η. It defines in the tangent P_m a P_{m-1}.

A set of m directions such that any $m-1$ of them determine the P_{m-1} so related to the remaining one we shall call a conjugate set of directions with respect to the direction σ. If we have an m-ple of curves on V_m such that their directions at every point form a conjugate set of directions with respect to the direction σ, then we say that we have a conjugate m-ple with respect to the direction σ.

The above P_{m-1} may be obtained in another way. If we take consecutive tangent P_{n-1}'s normal to σ and in the direction η, a point of intersection is defined by

$$\tau \cdot d\sigma = 0,$$

which can easily be reduced to the form

$$\tau \cdot \varphi \cdot \eta = 0$$

or

This P_{m-1} is the locus of points of intersection of consecutive tangent P_{n-1}'s normal to σ in the direction η.

Finally, the lines of curvature with respect to the direction σ are the only conjugate orthogonal m-ple with respect to the direction.

For let us assume an orthogonal conjugate m-ple and let η_i be unit vectors in these directions. Then

$$\varphi = \sum_{ij} \sigma \cdot I_{ij} \eta_i \eta_j.$$

Since η_i and η_j are conjugate, $\eta_i \cdot \varphi \cdot \eta_j = 0$, whence $\sigma \cdot I_{ij} = 0$, $i \neq j$, and this is given in (22) as the condition that the lines of curvature are the coordinate m-ple.

8. The osculating P_{m+1}

We are led to the direction h in another way which is of interest. Any point in the tangent P_m is given by

$$y = x + \sum_r \mu^r x_r,$$

in which the μ_r are parameters. Likewise if σ, as before, is a unit vector perpendicular to the variety, any point in the P_{m+1} determined by it and the tangent P_m is given by

$$y = x + \sum_r \mu^r x_r + \mu\sigma. \tag{23}$$

Now take any point z of P_n not in this P_{m+1}. There is through z a unique P_{n-m-1} of which every direction is perpendicular to every direction of the P_{m+1}. It intersects the P_{m+1} in just one point, which is then the projection of the given point on the P_{m+1}. We may thus project our variety on the P_{m+1}. Then in the P_{m+1} there is a unique direction normal to the projection of our variety, and hence the curvature of the projection may be defined in the usual manner. When this is done it turns out to be precisely the quantity that we have found as the curvature of the variety with respect to the direction σ.

The work is as follows.

The equations of the P_{n-m-1} are

$$(z-y)\cdot x_r = 0, \qquad (z-y)\cdot\sigma = 0,$$

and to find the point of intersection with the P_{m+1} we substitute the value of y from (23) in these and solve for μ, getting

$$\mu = (z-x)\cdot\sigma.$$

Since σ is a unit vector perpendicular to P_m, the perpendicular distance from the projection of z to P_m is μ.

Now let z be a neighboring point of the variety

$$z = x + dx + \tfrac{1}{2}d^2x.$$

Then its projection is a neighboring point of the projected variety and the distance of its projection from P_m becomes, by the above formula for μ and the fact that $\sigma\cdot dx = 0$, equal to

$$\mu = \tfrac{1}{2}\sum_{rs}\sigma\cdot x_{rs}\,du_r\,du_s.$$

Hence the curvature of the projected variety is

$$K = \frac{2\mu}{ds^2} = \sum_{rs} \sigma \cdot x_{rs} \lambda^r \lambda^s, \tag{24}$$

which is just the form we deduced from (19).

We now may ask, of all P_{m+1}'s through the tangent P_m which fits the variety best? If we define best fit by saying that the mean curvature of the projection of the variety on the P_{m+1} shall be a maximum, we find that the answer is that P_{m+1} which is determined by the vector h. This P_{m+1} we shall call the osculating P_{m+1} to the given variety.

One can go ahead and define similarly an osculating P_{m+2}, but that would involve third derivatives and will not be considered in this paper.

We remember that the normal curvature of the variety in the direction λ_h^r is I_{hh}, and by solving (15) we get

$$I_{hh} = \sum_{rs} x_{rs} \lambda_h^r \lambda_h^s.$$

Also by (24) the curvature of the projection of a curve having this direction, on the P_{m+1} through the tangent line defined by the direction λ_h^r, is

$$K = \sum_{rs} \sigma \cdot x_{rs} \lambda_h^r \lambda_h^s = \sigma \cdot I_{hh},$$

and since the projection of I_{hh} on the direction σ is $\sigma \cdot I_{hh}$ we have the theorem:

The projection of the curvature at the point of tangency of any curve of the variety on any P_{m+1} through the tangent P_m is equal to the curvature of the projection of the curve.

9. Canonical axes for the principal normal space

The principal curvatures with regard to the direction σ are the roots of (21). If we write this out as a polynomial in ϱ, and make the coefficient of the highest power of ϱ unity, then the other coefficients are the invariants that correspond to mean and total curvature in surface theory. We have already made use of the coefficient of ϱ^{m-1} in our mean curvature. Its maximum gave us the direction h. We shall now make use of the coefficient of ϱ^{m-2} which is the sum of the products of the principal curvatures with respect to the direction σ taken two at a time. The other coefficients, for whose simple treatment some elements of a theory of polyadics is desirable, will not be discussed in this paper. We now ask for what directions σ has this coefficient of ϱ^{m-2} stationary values? We find that there are $m(m+1)/2$ such directions, all mutually perpendicular. The work is as follows:

Denote the coefficient in question by D_2. It may be written

$$D_2 = \tfrac{1}{4}\sum_{ij\,ke}\begin{vmatrix} \sigma\cdot x_{ij} & \sigma\cdot x_{ie} \\ \sigma\cdot x_{kj} & \sigma\cdot x_{ke}\end{vmatrix}\cdot\begin{vmatrix} a^{ij} & a^{ie} \\ a^{kj} & a^{ke}\end{vmatrix}. \tag{25}$$

If this is expanded and the values of x_{rs} in terms of the I_{hk} are substituted,

$$D_2 = \tfrac{1}{2}\left\{\Big[\sum_h \sigma\cdot I_{hh}\Big]^2 - \sum_{hk}(\sigma\cdot I_{hk})^2\right\}$$

is obtained. This in turn may be written

$$D_2 = \sigma\cdot\psi\cdot\sigma,$$

in which ψ is the dyadic

$$2\psi = m^2 hh - \sum_{hk} I_{hk} I_{hk} = \sum_{hk}(I_{hh} I_{kk} - I_{hk} I_{hk}). \tag{26}$$

To find the stationary values of D_2 we have

$$D_2 = \sigma\cdot\psi\cdot\sigma \text{ and } \sigma\cdot\sigma = 1$$

and differentiating these we get

$$\sigma\cdot\psi\cdot d\sigma = 0 \text{ and } \sigma\cdot d\sigma = 0,$$

from which $d\sigma$ must be eliminated. The second equation tells us that we must choose $d\sigma$ so that it is perpendicular to σ while the first says that for any such choice the vector $\sigma\cdot\psi$ must be perpendicular to $d\sigma$. Hence the vectors $\sigma\cdot\psi$ and σ must have the same direction, or

$$\sigma\cdot\psi = k\sigma. \tag{27}$$

This defines in the principal normal space the axes of the hyperquadric

$$\sigma\cdot\psi\cdot\sigma = \text{const.}$$

From (27), by multiplying by σ, we find

$$\sigma\cdot\psi\cdot\sigma = k,$$

so that the values of k are the stationary values of D_2.

Now set

$$\sigma = \sum_{rs}\sigma_{rs} I'_{rs},$$

in which I'_{rs} are the polar set of vectors I_{rs}, i. e., I'_{ij} is in the space determined by the I's and is perpendicular to all the I's except I_{ij}, and

$$\begin{aligned} I'_{rs} \cdot I_{tu} &= 0 \qquad (r, s \neq t, u), \\ I'_{rs} \cdot I_{rs} &= \tfrac{1}{2} \qquad (r \neq s), \\ I'_{rr} \cdot I_{rr} &= 1. \end{aligned}$$

Multiplying this relation by I_{rs} we get

$$\sigma_{rs} = \sigma \cdot I_{rs},$$

and substituting these in (27) gives

$$\sum_{rs} \sigma_{rs} I'_{rs} \cdot \psi \cdot I_{tu} - k\sigma_{tu} = 0, \tag{28}$$

so that the values of k are given by the equation

$$| I'_{rs} \cdot \psi \cdot I_{tu} - k\varepsilon_{rs,tu} | = 0, \qquad r \leqq s,\ t \leqq u, \tag{29}$$

where

$$\begin{aligned} \varepsilon_{rs,tu} &= 0 \qquad (r, s \neq t, u), \\ \varepsilon_{rs,rs} &= \tfrac{1}{2} \qquad (r \neq s), \\ \varepsilon_{rr,rr} &= 1. \end{aligned}$$

We can find the sum of the k's by expanding this and taking the coefficient of the second highest power of k, and this gives

$$\sum_{r \leqq s} k_{rs} = \sum_{r \leqq s} I'_{rs} \cdot \psi \cdot I_{rs},$$

from which we easily deduce

$$\sum_{r \leqq s} k_{rs} = \tfrac{1}{2} G = \psi_s = \tfrac{1}{2} \sum_{rstu} a^{us}\, a^{rt}\, (ur, st). \tag{30}$$

This is called the Gaussian curvature of the variety. It is to be noted that it is the scalar of the dyadic ψ.

The directions σ defined by equation (28) form a mutually orthogonal system, which may be used as a canonical system of axes for the principal normal space.

10. The tangent P_m

We may write the unit tangent P_m as

$$P_m = \eta_1 \times \eta_2 \times \eta_3 \times \cdots \times \eta_m,$$

whence

$$dP_m = \sum d\eta_1 \times \eta_2 \times \cdots \times \eta_m,$$

but

$$d\eta_i = \sum_r \eta_{i,r}\, du_r = \sum_{r,j} I_{ji}\, \lambda_{jr}\, du_r + \sum_{rjk} \gamma_{ijk}\, \lambda_{kr}\, \eta_j\, du_r$$

in which we must remember that* $\gamma_{hhc} = 0$.

* *Absolute Calculus*, p. 149.

And so we have

$$dP_m = \sum_{rj} (\sum I_{j1} \times \eta_2 \times \cdots \times \eta_m)\, \lambda_{jr}\, du_r$$

since the latter part drops, due to having two identical η's in each term. Now

$$dx = \sum_r x_r\, du_r$$

and

$$\eta_h = \sum_r \lambda_h^r\, x_r,$$

and solving this

$$x_s = \sum_h \lambda_{hs}\, \eta_h,$$

whence

$$dx = \sum_{s,h} \lambda_{hs}\, \eta_h\, du_s.$$

So finally we get the formula

$$\begin{aligned} dx \times dP_m &= \sum_{shrj} (\sum \eta_h \times I_{j1} \times \eta_2 \times \cdots \times \eta_m)\, \lambda_{jr}\, \lambda_{hs}\, du_r\, du_s, \\ &= (-1)^{m-1} P_m \times \sum_{rs} x_{rs}\, du_r\, du_s, \\ &= (-1)^{m-1} P_m \times b. \end{aligned}$$

We are using the rules of n-dimensional vector algebra as given by Lewis.* From these rules one easily deduces the following relation between dot and cross products of two mth order vectors and one first order vector. The subscripts indicate order:

$$t_m \cdot (r_1 \times s_m) = (t_m \cdot s_m)\, r_1 + (r_1 \cdot t_m) \cdot s_m.$$

We dot-multiply P_m into our previous formula and apply this relation:

$$P_m \cdot (dx \times dP_m) = P_m \cdot (P_m \times b)\, (-1)^{m-1} = -P_m \cdot (b \times P_m),$$

$$(P_m \cdot dP_m)\, dx + (dx \cdot P_m) \cdot dP_m = -(P_m \cdot P_m)\, b - (b \cdot P_m) \cdot P_m.$$

Now

$$P_m \cdot dP_m = 0, \quad P_m \cdot P_m = 1, \quad P_m \cdot b = 0.$$

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* Gilbert N. Lewis, *On four-dimensional vector analysis, and its application in electrical theory*, Proceedings of the American Academy of Arts and Sciences, vol. 46 (1910), pp. 168—169.

Hence the above simplifies to

$$(dx \cdot P_m) \cdot dP_m = -b, \tag{31}$$

which is the natural extension of the similar formula of surface theory. The quantity $dx \cdot P_m$ is a vector of order $m-1$ in the tangent P_m and perpendicular to dx.

We have thus the expression of the vector second fundamental form in terms of the tangent plane and its differential.

Another formula that we want is that for $dP_m \cdot dP_m$. We have

$$dP_m \cdot dP_m = \Big[\sum_{rj}(\sum I_{j1} \times \eta_2 \times \cdots \times \eta_m)\,\lambda_{jr}\,du_r\Big]^2.$$

When this is multiplied out we have terms of the type

$$(\eta_1 \times \eta_2 \times \cdots \times \eta_{i-1} \times I_{ki} \times \eta_{i+1} \times \cdots \times \eta_m) \cdot (\eta_1 \times \eta_2 \times \cdots \times \eta_{j-1} \times I_{hj} \times \eta_{j+1} \times \cdots \times \eta_m),$$

which are zero unless the η missing from each factor is the same, in which case we have $I_{ki} \cdot I_{hi}$.

So

$$dP_m \cdot dP_m = \sum_{ihkrs} I_{hi} \cdot I_{ki}\,\lambda_{hr}\,\lambda_{ks}\,du_r\,du_s,$$

or

$$dP_m \cdot dP_m = \sum_{rs} U_{rs}\,du_r\,du_s, \tag{32}$$

in which

$$U_{rs} = \sum_{ihk} I_{hi} \cdot I_{ki}\,\lambda_{hr}\,\lambda_{ks} = \sum_{pq} a^{pq}\,x_{pr} \cdot x_{qs}$$

are invariants of the variety, which here have their geometric interpretation.

There is another set of invariants of the variety which we wish to use. They are

$$T_{rs} = \sum_{pq} a^{pq}\,x_{pq} \cdot x_{rs} = m\,h \cdot x_{pq}.$$

If now we form

$$\sum_{rs} T_{rs}\,du_r\,du_s = m\,h \cdot b, \tag{33}$$

we see from (19) that this is the curvature of the variety with respect to the direction h, or rather, a multiple of it. The relation $T_{rs} = 0$, $r \neq s$ is the condition that the principal m-ple of curvature be taken for the parametric curves, but, of course, it cannot always be done.

11. The Einstein equations

If now we set

$$G_{rs} = U_{rs} - T_{rs},$$

we find that

$$G_{rs} = \sum_{pq} a^{pq}(rp, qs)$$

from the integrability conditions. But $G_{rs} = 0$ are precisely the Einstein equations for material free space. From the work above we have

$$\sum_{rs} G_{rs}\, du_r\, du_s = dP_m \cdot dP_m - mh \cdot b, \tag{34}$$

and the Einstein conditions mean that this expression on the left is zero for all directions in the variety. A variety for which this is true we shall call an Einstein variety. For such a variety, $\dfrac{dP_m \cdot dP_m}{ds^2} = \dfrac{mh \cdot b}{ds^2}$.

So we have the theorem:

An Einstein variety is a variety for which the square of the vector rate of change of the tangent P_m for any direction in the variety is equal to $1/(mh)$ times the normal curvature of the variety in that direction. In this statement h is the scalar of the vector h.

For the principal asymptotic directions (34) becomes

$$dP_m \cdot dP_m = \sum_{rs} G_{rs}\, du_r\, du_s$$

or the rate of turning of the tangent P_m along a principal asymptotic direction is

$$\left(\frac{dP_m}{ds}\right)^2 = \frac{\sum_{rs} G_{rs}\, du_r\, du_s}{\sum_{rs} a_{rs}\, du_r\, du_s}.$$

12. The indicatrix

The normal curvature of the variety in any direction is a vector in the normal P_{n-m} at the point in question. If we take the locus of the end of this vector for all directions in the variety emanating from the point, we get in the normal P_{n-m} a variety which we shall call the indicatrix of our variety V_m for the point.

To study this locus we shall introduce a variable orthogonal m-ple. Let

$$\mu_{kr} = \sum_{h} c_{hk} \lambda_{hr},$$

whence

$$\lambda_h^s = \sum_k c_{hk}\mu_k^s, \qquad \lambda_{hs} = \sum_k c_{hk}\mu_{ks};$$

and since this is an orthogonal transformation

$$\sum_h c_{hk}c_{he} = \begin{cases} 0 & k \neq e \\ 1 & k = e \end{cases} = \sum_h c_{kh}c_{eh}.$$

Solving (15) we obtain

$$I_{hk} = \sum_{rs} \lambda_h^r \lambda_k^s x_{rs},$$

whence

$$I_{hk} = \sum_{rsmn} c_{hm}c_{kn}\mu_m^n \mu_n^s x_{rs}$$

$$= \sum_{mn} c_{hm}c_{kn}J_{mn},$$

in which we have denoted by J_{mn} the corresponding invariants of the m-ple μ. Solving these last equations we get

$$J_{ij} = \sum_{hk} c_{hi}c_{kj}I_{hk}. \tag{35}$$

The vectors J_{ij} therefore lie in the plane space determined by the I_{ij}. If the I_{ij} are all independent vectors this space is of dimensionality $m + \frac{1}{2}m(m-1) = \frac{1}{2}m(m+1)$. We shall assume that this is the case, so that $\frac{1}{2}m(m-1)$ is less than $n - m$ in our discussion.

In this case there is a system of vectors I'_{hk} polar to the I_{hk}. There is also a set of vectors J'_{hk} polar to the J_{hk}. A little computation shows that the transformation from the I'_{hk} to the J'_{hk} is precisely (35).

From the relation (35) we see that the indicatrix is defined parametrically by the equation

$$J = \sum_{hk} c_h c_k I_{hk}, \text{ in which } \sum_h c_h^2 = 1. \tag{36}$$

If we wish to define it implicitly we shall find it convenient to make use of the system of vectors x'_{rs} polar to the system x_{rs}. We have

$$x'_{rs} = \sum_{pq} (x'_{rs} \cdot I_{pq})\, I'_{pq}$$

whence

$$x'_{rs} = \sum_{pq} \lambda_p^r \lambda_q^s I'_{pq}, \tag{37}$$

so that

$$I'_{pq} = \sum_{rs} x'_{rs}\lambda_{pr}\lambda_{qs}. \tag{38}$$

If we denote by ψ'_{ij} the determinant

$$\psi'_{ij} \equiv \begin{vmatrix} x'_{ii} & x'_{ij} \\ x'_{ji} & x'_{jj} \end{vmatrix}, \tag{39}$$

then by direct substitution

$$I_{hh} \cdot \psi'_{ij} \cdot I_{hh} = 0 \qquad (i, j = 1, 2, \cdots, m)$$

so that the indicatrix lies on all the quadratic hypercones

$$\sigma \cdot \psi'_{ij} \cdot \sigma = 0. \tag{40}$$

These, with the hyperplane

$$m h' \cdot \sigma = 1, \tag{41}$$

completely define it, and are equivalent to (36). In this, h' is the vector formed by substituting I'_{ij} for I_{ij} in the formula for h. The count of the dimensionality of the indicatrix from either of these representations gives $m - 1$, as is also obvious from the definition.

If we denote by ψ' the dyadic formed by substituting I'_{ij} for I_{ij} in the formula for ψ, we have

$$h' = \frac{2}{m-1} h \cdot \psi'. \tag{42}$$

The determinantal equation (21) may be multiplied by the non-vanishing determinants $|\lambda^r_q| \cdot |\lambda^s_p| = |a_{rs}|$ to give the form

$$|\sigma \cdot I_{hk} - \varrho\, \epsilon_{hk}| = 0, \qquad \epsilon_{hk} = \begin{cases} 0 & h \neq k \\ 1 & h = k \end{cases}. \tag{43}$$

From this form we get the D_i immediately in terms of the I_{ij}. These, as well as the corresponding equations in the I'_{ij}, which we shall denote by D'_i, are invariants of the orthogonal substitution (35). Now in the space determined by the I_{ij}

$$D'_1(\sigma) \equiv m\sigma \cdot h' = 1$$

is a hyperplane through the end of h and contains the indicatrix. It is in fact the equation (41). Also

$$D'_i(\sigma) = 0 \qquad (i = 2, 3, \cdots, m) \tag{44}$$

are cones of degrees from 2 to m with vertices at the variety point and each satisfied by $\sigma = I_{hh}$, so that the indicatrix lies on each of them. In particular

$$D'_2(\sigma) \equiv \sigma \cdot \psi' \cdot \sigma,$$

where ψ' is

$$2\psi' = \sum_{rstu} (a_{rs}\, a_{tu} - a_{rt}\, a_{su})\, x'_{rs}\, x'_{tu}.$$

13. Intersection of consecutive normal P_{n-m}'s

Consider any point in the normal P_{n-m}. It is given by the formula

$$y = x + \varrho\sigma,$$

in which σ has its previous meaning of any unit vector in the normal P_{n-m} and ϱ is a parameter. A point in the normal P_{n-m} for a neighboring point of the variety is

$$z = x + \delta x + (\varrho + \delta\varrho)(\sigma + \delta\sigma),$$

and if these two P_{n-m}'s intersect, then for their points of intersection

$$\delta x + \varrho\delta\sigma + \sigma\delta\varrho = 0,$$

neglecting terms of higher order. When all differentials are written as derivatives with respect to s, this becomes

$$\sum_r x_r\lambda^r + \varrho\sum_r \sigma_r\lambda^r + \sigma\sum_r \varrho_r\lambda^r = 0, \tag{45}$$

and we can eliminate the last term by dot-multiplying in x_s, thus getting

$$\sum_r a_{sr}\lambda^r + \varrho\sum_r \sigma_r \cdot x_s\lambda^r = 0.$$

And since σ is perpendicular to x_s we can write this in the form

$$\sum_r a_{sr}\lambda^r + \varrho\sum_r \sigma \cdot x_{sr}\lambda^r = 0,$$

which is simply another way of writing (20), this ϱ being the reciprocal of the one there used. Hence if any σ is given, equations (20) and (21) determine the directions in the variety for which consecutive normal P_{n-m}'s intersect in points of the line determined by σ. Equation (21) shows that there are m such points of intersection along the line determined by any direction σ.

If we set $\tau = \varrho\sigma$ the equation (21) becomes

$$|\tau \cdot x_{rs} - a_{rs}| = 0,$$

and if in this we substitute

$$x = mh' + r$$

the above reduces to

$$|r \cdot x_{rs}| = 0,$$

which may be written in terms of the invariants of the net as

$$|r \cdot I_{ij}| = 0. \tag{46}$$

The geometric interpretation of this is as follows. In case the space determined by the I_{ij} is the total normal P_{n-m}, the equation (46) represents a hypercone, so that the locus of the points of intersection of consecutive normal P_{n-m}'s is a hypercone of order m, lying in the normal P_{n-m} and having its vertex at the point $x + mh'$. This point is the point diametrically opposite the variety point P, on the hypersphere which is the inverse of the hyperplane (41) with respect to the unit hypersphere with P as center. In the more general case the locus is still defined by (46) and we have:

Adjacent normal P_{n-m}'s intersect in a P_{n-2m} which is perpendicular to the space defined by the I_{ij} and which contains in it an element of the hypercone (46).

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EXPANSIONS IN TERMS OF SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

FIRST PAPER: MULTIPLE FOURIER SERIES EXPANSIONS*

BY

CHESTER C. CAMP

1. PRELIMINARY DISCUSSIONS

Consider the equation

$$L(u) \equiv \sum_{i=1}^{p} \frac{\partial u}{\partial x_i} + \lambda u = 0 \tag{1}$$

and its adjoint

$$M(v) \equiv -\sum_{i=1}^{p} \frac{\partial v}{\partial x_i} + \lambda v = 0, \tag{2}$$

which satisfy the identity

$$v L(u) - u M(v) \equiv \sum_{i=1}^{p} \frac{\partial}{\partial x_i}(u, v) \equiv 0. \tag{3}$$

A particular solution of (1) may be written

$$\bar{u} = e^{-\lambda x_1} \varphi(x_1 - x_2, x_1 - x_3, \cdots, x_1 - x_p). \tag{4}$$

To find the general solution we try

$$u = v\bar{u}.$$

Equation (1) then becomes

$$v L(\bar{u}) + \bar{u} \sum_{i=1}^{p} \frac{\partial v}{\partial x_i} = 0.$$

---

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Thus v is determined by $\sum_{i=1}^{p} \frac{\partial v}{\partial x_i} = 0$, whose solution is

$$v = \psi(x_1 - x_2, x_1 - x_3, \cdots, x - x_p).$$

From this it is seen that the general solution is of the same form as $\bar{u}$.

Likewise the general solution of (2) is found to be

$$v = e^{\lambda x_1} \varphi(x_1 - x_2, x_1 - x_3, \cdots, x_1 - x_p). \tag{5}$$

Choosing the intervals from $-\pi$ to π for x_i, $i = 1, 2, \cdots, p$, and taking the p-fold integral of equation (3) between these limits, we obtain after reduction

$$\begin{aligned} &\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\{[u(\pi, t_1, t_2, \cdots, t_{p-1})v(\pi, t_1, t_2, \cdots, t_{p-1}) \\ &\qquad - u(-\pi, t_1, \cdots, t_{p-1})v(-\pi, t_1, \cdots, t_{p-1})] \\ &+[u(t_1, \pi, t_2, t_3, \cdots, t_{p-1})v(t_1, \pi, t_2, \cdots, t_{p-1}) \\ &\qquad - u(t_1, -\pi, t_2, t_3, \cdots, t_{p-1})v(t_1, -\pi, t_2, \cdots, t_{p-1})] + \cdots \\ &+[u(t_1, t_2, \cdots, t_{p-1}, \pi)v(t_1, t_2, \cdots, t_{p-1}, \pi) - u(t_1, \cdots, \\ &\qquad t_{p-1}, -\pi)v(t_1, \cdots, t_{p-1} - \pi)]\}\, dt_1 \cdot dt_2 \cdots dt_{p-1} \equiv 0. \end{aligned} \tag{6}$$

If now we take as the boundary conditions for (1) the set

$$\begin{aligned} &L_1(u) \equiv u(-\pi, t_1, t_2, \cdots, t_{p-1}) - u(\pi, t_1, t_2, \cdots, t_{p-1}) = 0, \\ &L_2(u) \equiv u(t_1, -\pi, t_2, \cdots, t_{p-1}) - u(t_1, \pi, t_2, \cdots, t_{p-1}) = 0, \\ &L_3(u) \equiv u(t_1, t_2, -\pi, t_3, \cdots, t_{p-1}) - u(t_1, t_2, \pi, t_3, \cdots, t_{p-1}) = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &L_p(u) \equiv u(t_1, \cdots, t_{p-1}, -\pi) - u(t_1, \cdots, t_{p-1}, \pi) = 0, \end{aligned} \tag{7}$$

then a similar set in v may be taken for the adjoint system, since for such a choice equation (6) is satisfied for all values of t_i, $i = 1, 2, \cdots, (p-1)$.

A value of λ for which $L(u) = 0$ and $L_i(u) = 0$, $i = 1, 2, \cdots, p$, will also be a principal parameter value for the adjoint system. If a particular solution (4) $\not\equiv 0$ is found for which there is a discrete set of principal parameter values, that set so obtained will contain all the principal values for the general solution. To prove this, substitute $u = e^{-\lambda x_1}\varphi(x_1 - x_2, x_1 - x_3, \cdots,$

$x_1 - x_p$) in the boundary conditions $L_i(u) = 0$, $i = 1, 2, \cdots, p$. The result is, upon dividing all but the first by $e^{-\lambda t_1}$,

$$
\begin{aligned}
&e^{\lambda\pi}\varphi(-\pi - t_1, -\pi - t_2, \cdots, -\pi - t_{p-1}) \\
&\qquad - e^{-\lambda\pi}\varphi(\pi - t_1, \pi - t_2, \cdots, \pi - t_{p-1}) = 0, \\
&\varphi(t_1 + \pi,\ t_1 - t_2,\ t_1 - t_3, \ldots, t_1 - t_{p-1}) \\
&\qquad - \varphi(t_1 - \pi,\ t_1 - t_2,\ t_1 - t_3, \cdots, t_1 - t_{p-1}) = 0, \\
(8)\quad &\varphi(t_1 - t_2,\ t_1 + \pi,\ t_1 - t_3, \ldots, t_1 - t_{p-1}) \\
&\qquad - \varphi(t_1 - t_2,\ t_1 - \pi,\ t_1 - t_3, \cdots, t_1 - t_{p-1}) = 0, \\
&\cdots\cdots\cdots

and that the principal parameter values are

$$\lambda = ni \qquad (n = 0, \pm 1, \pm 2, \cdots, i = \sqrt{-1}).$$

This will be confirmed later by another method.

2. The formal expansion

If we consider two distinct principal values of λ, say λ_i and λ_j, and the corresponding principal solutions u_i and u_j for a particular φ, also the adjoint solutions, v_i and v_j, for the same φ, evidently

$$v_j L(u_i) - u_i M(v_j) = \sum_{k=1}^{p} \frac{\partial}{\partial x_k}(u_i v_j) + (\lambda_i - \lambda_j) u_i v_j. \tag{9}$$

Let us form the p-fold integral of this as we did for equation (3). Reducing as before and using the boundary conditions $L_\varkappa(u_i) = 0$, $\varkappa = 1, 2, \cdots, p$, and the adjoint boundary conditions $L_\varkappa(v_j) = 0$, $\varkappa = 1, 2, \cdots, p$, we obtain

$$(\lambda_i - \lambda_j) \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} u_i v_j \, dx_1 \, dx_2 \cdots dx_p = 0.$$

A division by $\lambda_i - \lambda_j$ gives the so called biorthogonality condition

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} u_i v_j \, dx_1 \, dx_2 \cdots dx_p = 0. \tag{10}$$

Since $\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} u_i v_i \, dx_1 \cdots dx_p = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \varphi^2 \, dx_1 \cdots dx_p > 0$, we can formally determine the coefficients c_n for the expansion

$$f(x_1, x_2, \cdots, x_p) = \sum_{n=-\infty}^{\infty} c_n u_n(x_1, x_2, \cdots, x_p), \tag{11}$$

i. e.,

$$c_n = \frac{\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(x_1, \cdots, x_p) \, v_i(x_1, \cdots, x_p) \, dx_1 \cdots dx_p}{\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} [\varphi(x_1, \cdots, x_p)]^2 dx_1 \cdots dx_p}. \tag{12}$$

The important question is whether the series in equation (11) actually converges to the value $f(x_1, x_2, \cdots, x_p)$. This is answered by a powerful method due to Birkhoff,* in which he uses contour integrals. Before employing this

* These Transactions, vol. 9 (1908), No. 4, pp. 377–395.

it is convenient to make the problem in the partial differential system depend upon p ordinary differential systems.

3. Dependence of the partial differential system on ordinary differential equations with boundary conditions

By putting

$$u = \prod_{i=1}^{p} \varrho_i(x_i), \qquad v = \prod_{i=1}^{p} \gamma_i(x_i), \tag{13}$$

in equations (1) and (2) and dividing by u and v respectively, we get

$$\sum_{i=1}^{p} \frac{\varrho_i'}{\varrho_i} + \lambda = 0, \qquad -\sum_{i=1}^{p} \frac{\gamma_i'}{\gamma_i} + \lambda = 0, \tag{14}$$

in which a prime indicates the derivative with respect to the one variable involved.

Clearly $\frac{\varrho_1'}{\varrho_1} + \lambda = -\sum_{i=2}^{p} \frac{\varrho_i'}{\varrho_i} = \text{const.} \equiv \mu_1$, say, and if we put $-\frac{\gamma_1'}{\gamma_1} + \lambda = \sum_{i=2}^{p} \frac{\gamma_i'}{\gamma_i} = \mu_1$, we shall have a second equation adjoint to the first.

Similarly we write in general

$$\frac{\varrho_x'}{\varrho_x} + \mu_{x-1} = -\sum_{i=x+1}^{p} \frac{\varrho_i'}{\varrho_i} = \mu_x, \qquad -\frac{\gamma_x'}{\gamma_x} + \mu_{x-1} = \sum_{i=x+1}^{p} \frac{\gamma_i'}{\gamma_i} = \mu_x \qquad (x = 2, 3, \cdots, (p-1)) \tag{15}$$

and finally

$$\frac{\varrho_p'}{\varrho_p} + \mu_{p-1} = 0, \qquad -\frac{\gamma_p'}{\gamma_p} + \mu_{p-1} = 0. \tag{16}$$

By this substitution the boundary conditions become, after reduction,

$$\varrho_i(-\pi) = \varrho_i(\pi), \qquad \gamma_i(-\pi) = \gamma_i(\pi) \qquad (i = 1, 2, \cdots, p). \tag{17}$$

Solutions of these systems are readily found to be

$$\varrho_1 = e^{(\mu_1-\lambda)x_1}, \qquad \gamma_1 = e^{(\lambda-\mu_1)x_1},$$

$$\varrho_\varkappa = e^{(\mu_\varkappa-\mu_{\varkappa-1})x_\varkappa}, \qquad \gamma_\varkappa = e^{(\mu_{\varkappa-1}-\mu_\varkappa)x_\varkappa}, \qquad \text{for } \varkappa = 2, 3, \cdots, \quad (p-1),$$

and

$$\varrho_p = e^{-\mu_{p-1}x_p}, \qquad \gamma_p = e^{\mu_{p-1}x_p}.$$

There is no loss of generality in putting the multiplicative constants equal to unity since the equations are homogeneous and in the expansion (11) they would otherwise cancel out.

The principal parameter values for the ϱ-system are given by

$$\mu_1 - \lambda = \pm n_1 i, \qquad \mu_2 - \mu_1 = \pm n_2 i, \qquad \cdots, \qquad \mu_{p-1} - \mu_{p-2} = \pm n_{p-1} i,$$
$$\mu_{p-1} = \pm n_p i,$$

where the n_j have the values $0, 1, 2, \cdots$, and $i = \sqrt{-1}$.

From this it is evident that

$$\varrho_j(x^j) = e^{in_j x_j} \qquad (j = 1, 2, \cdots, p;\ i = \sqrt{-1};\ n_j = 0, \pm 1, \pm 2, \cdots).$$

Similarly $\gamma^j(x_j) = e^{-in_j x_j}$ for the same values of j and n_j.

Hence the values for λ found above are confirmed and the form of one solution of (1) and (7) is given explicitly by

$$u_{n_1 n_2 \ldots n_p} = e^{i\sum_{\varkappa=1}^{p} n_\varkappa x_\varkappa} \qquad (n_\varkappa = 0, \pm 1, \pm 2, \cdots). \tag{18}$$

The corresponding solution of the adjoint system is

$$v_{n_1 n_2 \ldots n_p} = e^{-i\sum_{\varkappa=1}^{p} n_\varkappa x_\varkappa} \qquad (n_\varkappa = 0, \pm 1, \pm 2, \cdots). \tag{19}$$

In both solutions the corresponding value of λ is given by

$$\lambda_{n_1 n_2 \ldots n_p} = -i\sum_{\varkappa=1}^{p} n_\varkappa. \tag{20}$$

It is important to notice that the forms of the solutions given in (18) and (19) are not unique but correspond to only one choice of the φ function. Although any function φ in p variables, which has a period of 2π in each, may be utilized, it is desirable in this paper to restrict ourselves to a consideration of the form given above, since it is this choice of φ which leads always to expansions in multiple Fourier series.

4. Convergence of the expansion in multiple Fourier series

To introduce the proof by the contour method it is thought best to give the results from Birkhoff's article* for the simple case

$$\frac{du}{dx}+\lambda u=0, \qquad -\frac{dv}{dx}+\lambda v=0, \tag{21}$$

$$u(\pi)-u(-\pi)=0, \qquad v(\pi)-v(-\pi)=0,$$

since the present case is made to depend on it.

The Green's function for this system is

$$G(x,s)=\begin{cases}\dfrac{e^{\lambda(s-\pi-x)}}{2\sinh\lambda\pi}, & \text{if } s>x,\\[2ex] \dfrac{e^{\lambda(s+\pi-x)}}{2\sinh\lambda\pi}, & \text{if } s<x.\end{cases} \tag{22}$$

Its residue for the principal value $\lambda=\lambda_i$ is

$$R(x,s;\lambda_i)=\frac{u_i(x)v_i(s)}{\int_{-\pi}^{\pi}u_i(x)v_i(x)\,dx}. \tag{23}$$

The general term of the formal expansion for $f(x)$ is

$$\frac{u_i(x)\int_{-\pi}^{\pi}f(s)v_i(s)\,ds}{\int_{-\pi}^{\pi}u_i(s)v_i(s)\,ds}. \tag{24}$$

This is equal to

$$\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma_i}\int_{-\pi}^{\pi}G(x,s;\lambda)f(s)\,ds\,d\lambda, \tag{25}$$

* Loc. cit.

in which Γ_i is a contour in the λ-plane enclosing just the one simple pole λ_i of G. The sum of n terms of the expansion is given by

$$\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma}\int_{-\pi}^{\pi} G(x,s,\lambda)f(s)\,ds\,d\lambda, \tag{26}$$

where Γ is a contour containing $\lambda_1, \lambda_2, \cdots, \lambda_n$ and no other λ_i.

By assuming a sequence of circles Γ_n with centers at the origin, such that their distance from the nearest poles, ni and $-ni$, where $i=\sqrt{-1}$ and n is an integer, is at least $d>0$, the value of the series is found as the limit of the contour integrals, as $n\to\infty$.

By breaking the inner integral into two, i.e., from $-\pi$ to x and from x to π, and integrating as to s by parts, the integral is evaluated. If one uses the more convenient form

$$G(x,s;\lambda)=\frac{e^{\lambda(s-x)}}{2}\left[\operatorname{sgn}(x-s)+\coth\lambda\pi\right], \tag{27}$$

the two parts are quite similar. The steps may be sketched as follows:

$$\begin{aligned} I_n &= \frac{1}{2\pi i}\int_{\Gamma_n}\int_{-\pi}^{x}\frac{e^{\lambda(s-x)}}{2}(1+\coth\lambda\pi)f(s)\,ds\,d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_n}\left\{\left[\frac{e^{\lambda(s-x)}}{2\lambda}(1+\coth\lambda\pi)f(s)\right]_{-\pi}^{x}+O\left(\frac{1}{\lambda^2}\right)\right\}d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_n}\frac{1+\coth\lambda\pi}{2\lambda}f(x-0)\,d\lambda+O\left(\frac{1}{\lambda}\right) \\ &= \frac{f(x-0)}{4\pi i}\int_{\Gamma_n}\frac{d\lambda}{\lambda}+O\left(\frac{1}{\lambda}\right)=\frac{f(x-0)}{2}+O\left(\frac{1}{\lambda}\right) \end{aligned} \tag{28}$$

since

$$\int_{\Gamma_n}\frac{\coth\lambda\pi}{\lambda}\,d\lambda=O\left(\frac{1}{\lambda}\right).$$

Similarly the other part yields $\frac{f(x+0)}{2}+O\left(\frac{1}{\lambda}\right)$, and the limit of their sum gives $\frac{f(x-0)+f(x+0)}{2}$ uniformly for x, provided $f'(x)$ is continuous.

Let us see how this may be applied to the proof of convergence of the series (11). If we denote

$$\lambda - \mu_1, \quad \mu_1 - \mu_2, \quad \mu_{\varkappa-1} - \mu_\varkappa, \quad \mu_{p-1} \text{ where } \varkappa = 3, 4, \cdots, (p-1),$$

by $\nu_1, \nu_2, \nu_\varkappa, \nu_p$, respectively, then the Green's functions for the ordinary differential systems may be written

$$(29) \quad G_\varkappa(x_\varkappa, s_\varkappa, \nu_\varkappa) = \frac{e^{\nu_\varkappa(s_\varkappa - x_\varkappa)}}{2}[\operatorname{sgn}(x_\varkappa - s_\varkappa) + \coth \nu_\varkappa \pi], \varkappa = 1, 2, \cdots, p.$$

One may designate the residue of $G_\varkappa$ for the principal value $\nu_\varkappa = \nu_{\varkappa j}$ by

$$(30) \qquad R_{\varkappa j}(x_\varkappa, s_\varkappa, \nu_{\varkappa j}) = \frac{\varrho_{\varkappa j}(x_\varkappa)\gamma_{\varkappa j}(s_\varkappa)}{\int_{-\pi}^{\pi}\varrho_{\varkappa j}(x_\varkappa)\gamma_{\varkappa j}(x_\varkappa)dx_\varkappa} (\varkappa = 1, 2, \cdots, p).$$

The general term of our formal expansion (11) will then be

$$\frac{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} f v_j \, dx_1 \, dx_2 \cdots dx_p}{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} u_j v_j dx_1 \, dx_2 \cdots dx_p} u_j,$$

and the series may be represented by

$$\sum_{j=-\infty}^{\infty} \frac{\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} f \prod_{\varkappa=1}^{p}\gamma_{\varkappa j} dx_1 \cdots dx_p}{\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\prod_{\varkappa=1}^{p}\varrho_{\varkappa j}\prod_{\varkappa=1}^{p}\gamma_{\varkappa j} dx_1 \cdots dx_p} \prod_{\varkappa=1}^{p}\varrho_{\varkappa j}(x_\varkappa) \equiv F(x_1, \cdots, x_p).$$

Then, since

$$G_\varkappa(x_\varkappa, s_\varkappa; \nu_{\varkappa j}) = \frac{\varrho_{\varkappa j}(x_\varkappa)\gamma_{\varkappa j}(s_\varkappa)}{(\nu_\varkappa - \nu_{\varkappa j})\int_{-\pi}^{\pi}\varrho_{\varkappa j}(x_\varkappa)\gamma_{\varkappa j}(x_\varkappa)dx_\varkappa} + \sigma_\varkappa(x_\varkappa, s_\varkappa; \nu_\varkappa)$$
$$(\varkappa = 1, 2, \cdots, p),$$

where $\sigma_\varkappa$ is analytic in $\nu_\varkappa$ at $\nu_\varkappa = \nu_{\varkappa j}$, if we define the residue of p complex variables $z_\varkappa$, $\varkappa = 1, 2, \cdots, p$, as the coefficient of $\prod_{\varkappa=1}^{p}\frac{1}{(z_\varkappa - z_{\varkappa j})}$, then the

value of $\prod_{\varkappa=1}^{p} G_\varkappa(z_\varkappa)$ for $z_\varkappa = z_{\varkappa j}$ is

$$\frac{1}{(2\pi i)^p}\int_{\Gamma_p}\int_{\Gamma_{p-1}}\cdots\int_{\Gamma_1}\prod_{\varkappa=1}^{p}\frac{G_\varkappa(z_\varkappa)}{(z_\varkappa - z_{\varkappa j})}\,dz_1\,dz_2\cdots dz_p,$$

and that of the term of F for which $\nu_\varkappa = \nu_{\varkappa j}$, $\varkappa = 1, 2, \cdots, p$, will be the same as that of

$$(31)\quad \frac{1}{(2\pi i)^p}\int_{\Gamma_p}\cdots\int_{\Gamma_1}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} f(s_1, \cdots, s_p)\prod_{\varkappa=1}^{p}[G_\varkappa(x_\varkappa, s_\varkappa; \nu_\varkappa)\,ds_\varkappa\,d\nu_\varkappa]$$

where $\Gamma_\varkappa$ refers to a contour in the $z_\varkappa$-plane, (or $\nu_\varkappa$-plane), containing the simple pole $z_{\varkappa j}$, (or $\nu_{\varkappa j}$), because terms of the form $\sigma_\varkappa(x_\varkappa, s_\varkappa; \nu_\varkappa)\Big/\prod_{\substack{i=1\\ i\neq\varkappa}}^{p}(\nu_i - \nu_{ij})$ will contribute nothing.

Hence if F converges, it will be represented by the limit as $\nu_\varkappa \to \infty$, $\varkappa = 1, 2, \cdots, p$, oft he same integral (31) taken around sequences of circular contours in the $\nu_\varkappa$-planes, which are drawn in these several planes as that for the simple problem was drawn in the λ-plane.

Let us now break up the integral (31) into 2^p pieces by taking every combination formed by dividing each real interval from $-\pi$ to π into two intervals of the form $-\pi$ to $x_\varkappa$ and $x_\varkappa$ to π, for $\varkappa = 1, 2, \cdots, p$.

Consider the piece

(32)

$$\frac{1}{(2\pi i)^p}\int_{\Gamma_p}\cdots\int_{\Gamma_1}\int_{-\pi}^{x_p}\cdots\int_{-\pi}^{x_1} f(s_1, \cdots, s_p)\prod_{\varkappa=1}^{p}\left[\frac{e^{\nu_\varkappa(s_\varkappa - x_\varkappa)}}{2}(1+\coth\nu_\varkappa\pi)\,ds_\varkappa\,d\nu_\varkappa\right].$$

Integrating by parts as before p times, we obtain

$$\frac{1}{(2\pi i)^p}\int_{\Gamma_p}\cdots\int_{\Gamma_1}\frac{f(x_1-0, \cdots, x_p-0)}{2^p}\prod_{\varkappa=1}^{p}\left[(1+\coth\nu_\varkappa\pi)\frac{d\nu_\varkappa}{\nu_\varkappa}\right]$$
$$+ O\left(\frac{1}{\nu_p}\right) = \frac{f(x_1-0, x_2-0, \cdots, x_p-0)}{2^p} + O\left(\frac{1}{\nu_p}\right).$$

Similarly the piece whose upper limits are π and whose lower limits are $x_p, x_{p-1}, \cdots, x_1$, respectively, has the value

$$I^{\pi, \pi, \cdots, \pi}_{x_p, x_{p-1}, \cdots, x_1} \equiv \frac{f(x_1+0, x_2+0, \cdots, x_p+0)}{2^p} + 0\left(\frac{1}{\nu_p}\right)$$

and it is clear that the limit of F is given by

$$\sum \frac{f(x_1 \pm 0, x_2 \pm 0, \cdots, x_p \pm 0)}{2^p},$$

in which the summation includes every possible combination formed by choosing just one sign for each argument of f. We may therefore state the

THEOREM. *Let $f(x_1, x_2, \cdots, x_p)$ be made up of a finite number of pieces in the region $-\pi \leqq x_\varkappa \leq \pi$, $\varkappa = 1, 2, \cdots, p$, each real, continuous, and possessing continuous partial derivatives. The multiple Fourier expansion connected with the partial differential equation $\sum_{\varkappa=1}^{p} \frac{\partial u}{\partial x_\varkappa} + \lambda u = 0$, and the boundary conditions $L_\varkappa(u) = 0$ $(\varkappa = 1, 2, \cdots, p)$, namely, the series*

$$F \equiv \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_p=-\infty}^{\infty} \frac{1}{(2\pi)^p} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(s_1, \cdots, s_p) \prod_{\varkappa=1}^{p} (e^{-in_\varkappa(s_\varkappa - x_\varkappa)}\, ds_\varkappa),$$

converges to $\frac{1}{2^p} \sum f(x_1 \pm 0, x_2 \pm 0, \cdots, x_p \pm 0)$ for each point of the region $-\pi \leqq x_\varkappa \leqq \pi$ $(\varkappa = 1, 2, \cdots, p)$, provided we interpret the argument $(\pm\pi + 0)$ at points of the boundary to mean $(-\pi)$ and the argument $(\pm\pi - 0)$ to mean (π). In any subregion in which f is continuous and possesses continuous partial derivatives the series converges uniformly to f.

From the fact that the function to be expanded is real and that the solution $u = e^{\sum_{\varkappa=1}^{p} n_\varkappa i x_\varkappa}$ may be written

$$u = \prod_{\varkappa=1}^{p} (\cos n_\varkappa x_\varkappa + i \sin n_\varkappa x_\varkappa)$$

it is evident that F will contain, besides a possible constant, only terms of a multiple Fourier series. This may be illustrated by the following.

Example. Expand $f(x, y, z) = xyz$ in a multiple Fourier series.

Here

$$f(x, y, z) = \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_{lmn} e^{lix + miy + niz}$$

where

$$c_{lmn} = \frac{1}{8\pi^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} xyz\, e^{-lix-miy-niz}\, dx\, dy\, dz$$

$$= \frac{i^3}{lmn}\cos l\pi\,(\cos m\pi)\cos n\pi.$$

Hence

$$xyz = \sum_{l=-\infty}^{\infty}\frac{i}{l}\cos l\pi\, e^{lix}\sum_{m=-\infty}^{\infty}\frac{i}{m}\cos m\pi\, e^{miy}\sum_{n=-\infty}^{\infty}\frac{i}{n}\cos n\pi\, e^{niz}$$

$$= \sum_{l=1}^{\infty}\left(-\frac{2}{l}\cos l\pi\sin lx\right)\sum_{m=1}^{\infty}\left(-\frac{2}{m}\cos m\pi\sin my\right)\sum_{n=1}^{\infty}\left(-\frac{2}{n}\cos n\pi\sin nz\right),$$

since the terms may be rearranged and the value of any of the c's with a zero subscript is zero. This series converges uniformly to xyz for any interior point and converges to zero on the boundary.

It is clear that the restrictions on f might be lightened and that its region might easily be extended.

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EULER ALGEBRA*

BY

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I. PRELIMINARY

1. **Introduction.** The majority of arithmetical functions specify either additive or multiplicative properties of numbers, ordinary or algebraic. For additive functions the appropriate analysis is the theory of integral power series; for multiplicative, the theory of Dirichlet series and Euler products. Of the last the simplest example is $\Pi(1-p^{-s}), |s| \geqq 1$, extending to all primes $p > 1$. A relation between arithmetical functions which can be so interpreted that it remains true when the arguments of all the functions are replaced by suitably defined elements other than numbers is called qualitative. Such relations are numerous. A simple example is: the totient of an integer is equal to the sum of the totients of all the divisors of the integer. A non-qualitative relation is called quantitative. In the derivation of either kind of relation infinite processes may or may not be used. If the former, then in the case of quantitative relations the convergence of the processes is essential, while in the case of qualitative relations it is irrelevant. Milder instances of the last assertion are sometimes said to be self-evidently true, in particular this: the results obtained by equating coefficients after formal manipulations of series are obviously valid, and it is unnecessary to examine the convergence of the series. Whether obvious or not the general assertion can be proved and is established incidentally in the course of this discussion.

In the investigation of qualitative relations an algebra E (Euler) emerges as the appropriate algorithm. The elements upon which E is ultimately based are abstract, that is, any marks subject to the formal laws of common algebra. This algebra E is in a sense the resultant of two much simpler algebras, C (Cauchy) and D (Dirichlet). C is the algebra of power series in one variable, and is equivalent to the Grassmann-Gibbs indeterminate-product algebra generated by a modulus and a single unit; D is the algebra of Dirichlet series and is the direct product of an infinite number of algebras C. C and D are necessary preliminaries for E. By means of E the qualitative properties above described receive a wide generalization. Numerous applications of E have

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been made, both to unify existing treatments of arithmetical functions and to obtain new results in profusion. For simplicity, and to preserve the continuity of the discussion, we omit illustrations and applications except where necessary to make clearer the nature of the abstract theory. For easy reference we put here some definitions upon which is based everything that follows.

(1) If the sum, difference, product and quotient (the divisor not being any one of certain elements, called excepted) of any two identical or distinct elements of a set is an element of the set, the set is called a system.

A system thus differs from an abstract field only by the exclusion of division by each of several elements instead of by one. It is assumed that each rational combination formed in accordance with (1) has a unique interpretation, and that there is at least one set of interpretations of all such combinations which is self-consistent.

(2) If with respect to the elements of a given system a rational operation is unspecified beyond the condition that it obeys the formal laws of common algebra, the operation is called formal, otherwise, special. A system is abstract or special according as none or at least one of the four rational operations is special.

For example, if the elements $a, b, \cdots$ of the system S are the ideals of a realm R, the combinations $a \pm b$ (not to be confused with $(a+b)$, $(a-b)$, the G. C. D. and L. C. M. of a, b) are formal, for the sum $a+b$ and the difference $a-b$ of two ideals are purely formal. That no interpretations in either S or R beyond the formal need be assigned $a \pm b$ to obtain results of importance in S, R is evident from established usage.

(3) The unit ϵ with respect to formal multiplication has the properties $\epsilon f = f = f\epsilon$, f any element of the system.

(4) Division by ϵ, multiplication, addition and subtraction being as in (1), the aggregate of elements derived by iterations of these operations from a given set of elements is called an annulus, which is abstract or special according as none or some of the above operations are special, and the set is its base.

An annulus thus differs from a system by the omission of division except by the multiplicative unit. Theorems valid for an annulus have therefore a greater extension than their analogues for a system. If the elements are algebraic numbers and the rational operations as in arithmetic, the definition of annulus becomes that of Ring (German), anneau (French).

(5) If some or all of the elements of a system (or annulus) can be arranged in a definite order by some rule, the arrangement is called a sequence. An element may occur more than once in a sequence.

(6) A sequence whose first element is zero is called exceptional.

(7) The notation $(a)_n$ is used for the sequence $a_0, a_1, a_2, \cdots, a_n, \cdots$ in which the suffixes of the successive elements run $0, 1, 2, \cdots, n, \cdots$, and the notation $\{a\}_n$ for the sequence $a_1, a_2, \cdots, a_n, \cdots$ in which the suffixes run

$1, 2, \cdots, n, \cdots$. By an obvious change in the notation of the elements any sequence may be written in either notation. It will be of assistance to observe that the $(a)_n$ notation is used exclusively with what are called C processes, the $\{a\}_n$ with D.

(8) In C, D the excepted elements are the exceptional sequences. In other words, division as defined presently for sequences in either algebra C, D is possible when and only when the divisor is a sequence whose first term is different from zero. This actually is a theorem rather than a definition, but we state it here for emphasis and clearness. Its proof is immediately obvious from the definitions of division in C, D.

(9) Two sequences $(a)_n, (b)_n$ are equal, $(a)_n = (b)_n$, when and only when $a_i = b_i$ $(i = 0, 1, 2, \cdots)$; and $\{a\}_n = \{b\}_n$ when and only when $a_i = b_i$ $(i = 1, 2, \cdots)$.

From the definition it will appear later that we might have taken instead of sequences as the elements from which to construct C, D, E, certain hypercomplex numbers.

It will not be necessary hereafter to emphasize that rational operations indicated by the usual notations of common algebra are formal as in (2) unless otherwise stated, for where they are special they are so indicated by the appropriate symbolism. The last is necessary in order to prevent confusion in reading formulas, for in all four distinct sets of meanings (including the formal) are assigned to the operations.

In all that follows it is assumed that the $a, b, c, \cdots$ with any suffixes are elements of some system. All proofs in the paper depend only upon the foregoing definitions of which (9) is the most frequently used. For certain developments it is advantageous, however, to replace (9) by (9.2), which comes from the following fundamental postulate (or definition):

(9.1) The *series* $\sum a_n \equiv a_0 + a_1 + \cdots + a_n + \cdots$ is uniquely determined by the sequence $(a)_n$ of its terms; or, *a series is uniquely determined by the sequence of its terms.*

We assert only that when the law of formation of the successive elements $a_0, a_1, \cdots, a_n, \cdots$ of the sequence is assigned, the terms, and in particular the general term, of the series are uniquely known. The indicated sum $a_0 + a_1 + \cdots + a_n + \cdots$ is merely the symbol obtained by writing the elements of the sequence in their given order with plus signs between them. In particular, if the elements $a_0, a_1,$ are numbers and addition is as in arithmetic it is immaterial under the definition (9.1) whether or not a "sum" of the series exists; if the elements are not numbers, "sum" is meaningless, "sum" here being understood in its customary sense in relation to series, viz., as one definite number which may consistently replace the series in numerical calculations. This sufficiently emphasizes that everywhere in this paper "series" has the meaning (9.1) and no other.

The symbolic equality $(a)_n \sim \sum a_n$ is to be real: the series determined by $(a)_n$ is $\sum a_n$; and we shall say that each of $(a)_n$, $\sum a_n$ is associated with the other.

(9.2) Two series $\sum a_n$, $\sum b_n$ are defined to be equal, $\sum a_n = \sum b_n$, when and only when their associates are equal, $(a)_n \sim \sum a_n$, $(b)_n \sim \sum b_n$, $(a)_n = (b)_n$.

Note that under this definition if the terms c_n are numbers and addition is as in arithmetic, and if $\sum c'_n$ is the series obtained by rearrangement of terms from the absolutely convergent series $\sum c_n$, we do not have $\sum c_n = \sum c'_n$.

(9.3) As usual in mathematical logic two relations are called formally equivalent if each implies the other.

The connections between the operations of the algebras C, D, E with operations upon series will be pointed out incidentally in the development of each algebra. The consequences of operations X ($X = C, D, E$) will be shown to be formally equivalent to the consequences of certain other operations upon series. Hence if we establish the validity of the first consequences, that of the second will follow, and as a convenience in manipulation we may when desired replace the first set of operations by the second.

It will be well here to give some indication of the origin of the following abstract theory which, although it has an interest as an example of an algebra isomorphic (except for division by more than one "zero") to common algebra yet widely different in content, is important chiefly for the uses to which it has been put. There exists a great mass of special theorems concerning arithmetical functions clustering about the unique factorization theorem. These have been stated and proved individually by various methods, the proofs in some cases, where they depend upon devices peculiar to the theory of numbers, degenerating to a succession of divinations without any unifying principle. In several instances it seems almost as if the results had first been found by means of obvious infinite processes and then translated for esthetic reasons into pure arithmetic. In working over this mass of results it was noticed that it could be unified, made symmetrical by extension in several directions, and finally reduced to an isomorph of common algebra by means of E. It will perhaps seem that such simple processes as those of E can lead to nothing less obvious than the relations from which we start. This is not the case. Even so trivial an identity as $\alpha = \alpha\beta/\beta$ when developed in E gives at once numerous valuable consequences, one of which is the totient theorem already cited, upon choosing for α, β the simplest rational functions of one or two variables t, ξ, such as $1 \pm \xi$, $1 \pm \xi^2$, $1 \pm t\xi$, $1 \pm t^2\xi^2$, etc. Many of these consequences have formed the subjects of notes or elaborate papers, summarized in volume 1, chapters V, X, XIX of Dickson's *History of the Theory of Numbers*. When the appropriate elementary algebra is used, the arithmetical relation often becomes as obvious as $\alpha = \alpha\beta/\beta$. One important application is outlined in § 12; many others, in complete detail, will be found in the paper referred to in § 11, footnote.

2. **Algebra** C. The C sum $(a + b)_n$ of $(a)_n$ and $(b)_n$ is the sequence $a_i + b_i$ ($i = 0, 1, \cdots$); the C difference $(a - b)_n$ is the sequence $a_i - b_i$ ($i = 0, 1, \cdots$)

and the C zero sequence $(0)_n$ is $(a_i - a_i)$, $(a)_n$ being any sequence. Obviously $(a+b)_n = (b+a)_n$, and if $(a+b)_n = (c)_n$, $(b+d)_n = (e)_n$, then $(c+d)_n = (a+e)_n = (a+b+d)_n = a_i + b_i + d_i$ $(i = 0, 1, \cdots)$. Hence C addition of sequence is commutative and associative, and similarly for subtraction. We shall denote the nth element $a_n + b_n + d_n$ of $(a+b+d)_n$ by ${}_n(a+b+d)$, and so in all like cases. Thus the nth element of $(a, b)_n$ next defined is ${}_n(a, b)$. The C product $(a, b)_n$ of $(a)_n$, $(b)_n$ is the sequence ${}_n(a, b)$ $(n = 0, 1, \cdots)$, where ${}_n(a, b) = a_n b_0 + a_{n-1} b_1 + \cdots + a_1 b_{n-1} + a_0 b_n$.

Proceeding similarly with $(a)_n$, $(b)_n, \cdots$, $(k)_n$ we obtain their C product $(a, b, \cdots, k)_n$ defined by ${}_n(a, b, \cdots, k) = \sum a_\alpha b_\beta \cdots k_\varkappa$, $\alpha + \beta + \cdots + \varkappa = n$, $0 \leqq \alpha, \beta, \cdots, \varkappa \leqq n$. The unit $(u)_n$ with respect to C multiplication is the sequence $u_0 = \varepsilon$ (cf. § 1 (3)), $u_j = 0, j > 0$. Obviously $(a+b, c)_n = (a, c)_n + (b, c)_n$, etc., and C multiplication is commutative, associative, and with respect to C addition and subtraction of sequences, distributive.

THEOREM I. *Any set of sequences, formed from the elements of a system, is the base of a special annulus in which the rational operations are C addition, subtraction and multiplication of sequences, and all the elements of the annulus are sequences.*

This is the C annulus to the given base. The number of elements (sequences) in the base may be either finite or infinite.

To connect this with series let t be a parameter which by definition has with respect to $a_i, b_j, c_k, \cdots$ all the formal properties of common algebra, so that $t^n a_i = a_i t^n$, etc. Beyond this $t^n a_i$ is undefined. For all integral values $\geqq 0$ of n write $\alpha_n = a_n t^n$, $\beta_n = b_n t^n, \cdots, \gamma_n = c_n t^n$, and put, according to (11),

$$(\alpha)_n \infty \sum a_n t^n, \qquad (\beta)_n \infty \sum b_n t^n, \qquad \cdots, \qquad (\gamma)_n \infty \sum c_n t^n.$$

Then from Theorem I it follows that for a definite choice of the signs, $(\pm\alpha \pm \beta \pm \cdots \pm \gamma) \infty (\pm a_n \pm b_n \pm \cdots \pm c_n) t^n$, $\sum_n (a, b, \cdots, c) t^n \infty (\alpha, \beta, \cdots, \gamma)_n \infty \sum a_n t^n \times \sum b_n t^n \times \cdots \times \sum c_n t^n$, where, as in all similar cases, the indicated operations upon the series are to be performed formally and the result is then to be rearranged as a power series in t.

If $(a)_n$ is not an exceptional sequence (§ 1(6)), and $(a, a')_n = (u)_n$, $(a')_n$ is called the C reciprocal of $(a)_n$, and we write $(a')_n = (u/a)_n$. From the given relation, ${}_0(a, a') = \varepsilon$, ${}_n(a, a') = 0$ $(n > 0)$, and hence $a'_n a_0 + a'_{n-1} a_1 + \cdots + a'_1 a_{n-1} + a'_0 a_n = 0$ $(n > 0)$; whence, with $a'_0 a_0 = \varepsilon$ we obtain the explicit form for $n > 0$,

$$a'_n = -\varepsilon (-a_0)^{-n} \begin{vmatrix} a_1 & a_2 & a_3 \cdots a_{n-2} & a_{n-1} & a_n \\ a_0 & a_1 & a_2 \cdots a_{n-3} & a_{n-2} & a_{n-1} \\ 0 & a_0 & a_1 \cdots a_{n-4} & a_{n-3} & a_{n-2} \\ \cdot & \cdot & \cdot \cdot \cdot \cdot \cdot & \cdot & \cdot \\ 0 & 0 & 0 \cdots a_0 & a_1 & a_2 \\ 0 & 0 & 0 \cdots 0 & a_0 & a_1 \end{vmatrix},$$

which with $a_0' = \varepsilon/a_0$ completely defines $(a')_n$. It has been sufficiently emphasized that the divisor $(a)_n$ in $(u/a)_n$ is never an exceptional sequence. Evidently the C product of $(d, u/b)_n$ and $(b)_n$ is $(d)_n$. Hence if $(a)_n = (d, u/b)_n$, then $(a, b)_n = (d)_n$; and conversely, since the C product of $(a, b)_n$ and $(u/b)_n$ is $(a)_n$, the equality $(a, b)_n = (d)_n$ implies $(a)_n = (d, u/b)_n$. Continuing thus, or directly from this result and Theorem I, we infer that, none of $(a), (b), \cdots, (c)$ being exceptional the relation $(a, b, \cdots, c, d, e, \cdots, f)_n = (k, l, \cdots, s)_n$ implies $(d, e, \cdots, f)_n = (k, l, \cdots, s, u/a, u/b, \cdots, u/c)_n$, or as we shall write it more suggestively $(d, e, \cdots, f)_n = \left(\frac{k, l, \cdots, s}{a, b, \cdots, c}\right)_n$.

THEOREM II. *The aggregate of all sequences formed from the elements of a system constitutes a special system in which the four rational operations are C addition, subtraction, multiplication and division, and the excepted elements are the exceptional sequences.*

The complicated determinant forms occurring in division are not required in practice; their only use is in demonstrating the existence of C reciprocals. The like applies to D. The series equivalent of C division being evident it need not be written.

By (9) the equality of two sequences is formally equivalent to the identity of their general terms, and Theorem II shows how identities of this kind may be obtained on starting from any given set of sequences. From the foregoing it is clear that the same identities would be reached if we operated with the associated series instead of the sequences as summarized in the following theorem. Hence (cf. the remarks after (9.3)) the formal use of the series is validated.

THEOREM II'. *If $R(x, y, \cdots, z) = 0$ is a rational relation in any system, and if, further, in algebra C the C relation $R((\alpha)_n, (\beta)_n, \cdots, (\gamma)_n) = 0$ holds, where $(\alpha)_n \sim \sum a_n t^n, (\beta)_n \sim \sum b_n t^n, \cdots, (\gamma)_n \sim \sum c_n t^n$, then the C relation is equivalent to $R((a)_n, (b)_n, \cdots, (c)_n) = 0$, which in turn is formally equivalent to $R(\sum a_n t^n, \sum b_n t^n, \cdots, \sum c_n t^n) = 0$ in the system and is obtained from the last by equating to zero the coefficient of t^n.*

An alternative point of view may be noted in passing. Let ε_n denote the sequence whose $(n+1)$th element is 1, and whose remaining terms are zeros. Then $(a)_n$ is $\sum_0^\infty a_n \varepsilon_n$, which (as in hypercomplex numbers) may be written a. Then algebra C is the Grassmann-Gibbs product algebra generated by a modulus (1) and a single unit e, $(\varepsilon_0 = 1, \varepsilon_n = e^n)$. This algebra is equivalent to that of power series in one variable.

As they are frequently useful we consider briefly the meanings to be assigned to irrational operations in C. Algebras D, E are similarly discussed in this respect with a few obvious changes in notation and consequent modifications in the interpretations of results, so it will be unnecessary to repeat

the argument for D, E. We are then to assign selfconsistent meanings to irrational functions of the elements, which are sequences, in C such that these are compatible with the rational functions.

If the a in $(a, a, \cdots, a)_n$ occurs precisely r times we write this C product $(a^r)_n$ and define it to be the rth power of $(a)_n$. Then if $(b)_n = (a^r)_n$ we call $(a)_n$ the rth root of $(b)_n$ and write $(a)_n = (b^{1/r})_n$. From this $(b^{s/r})_n$ is defined for r, s integers > 0 by $(b^{s/r})_n = (a^s)_n$. The C reciprocal $(u/a, u/a, \cdots, u/a)_n$ of $(a^r)_n$ is written $(a^{-r})_n$. Hence from the foregoing $(b^{-s/r})_n$ is defined and is equal to $(a^{-s})_n$, the notation being as above. Hence, t being any rational non-zero number, (a^t) is defined. The explicit forms of the elements in this sequence can be written easily, but as they are not required in applications we omit them. Incidentally the explicit forms prove the existence of such sequences. For fractional values of t they enter only as links in chains of transformations. It is readily seen from the definitions that when $t, s, \cdots$ are rational numbers, $(a^t)_n, (a^s)_n, \cdots$ are subject to the formal laws of common algebra.

THEOREM II″. *Algebra C in its complete form is abstractly identical with common algebra, and hence each identity or theorem in the latter has a unique dual in terms of elements of C, and this dual is expressible ultimately as an identity or theorem connecting the nth (general) elements of sequences related to one another by the operations of C.*

In practice we apply this as if the elements $(a)_n, (b)_n, \cdots$ were those of common algebra, $a, b, \cdots$. Thus we may ignore the special notations $(a)_n, (a, b)_n$, etc., and write simply $a, b, ab, a/b$, etc. In the results each $a, b \cdots$ is interpreted as the nth element of the sequence which it represents, and similarly for ab, etc., viz. ab is not the formal product of the nth elements of a, b, but is the nth element of the C product of a and b. As remarked, precisely similar considerations apply to D, E, and this exposition with a few slight changes can be fitted to them.

We shall reach our goal E through C via D. For E, C is fundamental. The point of departure for obtaining relations in E will be seen to be the identities obtained by applying Theorems II, II′. Hence we next indicate the manner in which these may be found. We recall first that in (9.1) any quantitive (numerical) significance of an infinite series was abandoned. Hence we discard also the concept of an analytic function and all of its consequences. Nevertheless we shall assign a meaning to the representation of a function of an element by an infinite series which is significant in all circumstances. In particular if the terms of the series are numbers and the operations as in arithmetic, the function is represented by the series whether or not it is summable by any method.

Let $\alpha, \beta, \cdots, \gamma, \varkappa, \lambda, \cdots, \sigma$ be any system of elements, and $(\alpha)_n, (\beta)_n, \cdots, (\sigma)_n$ sequences as usual. Note that α is not a member of $(\alpha)_n$, and similarly for the rest. Let $(\Delta)_n$ be a sequence of operators, φ *any* element such that the results of operating with $\Delta_n (n = 0, 1, \cdots)$ upon φ are each uniquely significant, and let the result $\Delta_n \varphi$ of operating with Δ_n upon φ be φ_n, or symbolically $\Delta_n \varphi = \varphi_n$. We assume that for m, n integers $\geqq 0$, $\Delta_m \varphi_n$

$= \Delta_m(\Delta_n\varphi) = \varphi_{m+n}$; so the elements (operators) of $(\Delta)_n$ obey with respect to suffixes the index laws of ordinary algebra, thus $\Delta_m\Delta_n = \Delta_{m+n} = \Delta_n\Delta_m$, etc. We shall say that $(\varphi)_n$ is developed by $(\Delta)_n$ from φ, and write this $(\Delta)_n\varphi = (\varphi)_n$. When there is no occasion to indicate the specific developer $(\Delta)_n$ we shall write φ' for $(\Delta)_n\varphi$. If $\varphi = \psi$, then by definition $\varphi' = \psi'$. For simplicity we assume that $\Delta_0\varphi \equiv \varphi_0$ is different from zero. Developers are classified according to their properties with respect to sums, differences, products and quotients of elements. The $(\Delta)_n$ considered now is called, for obvious reasons, a C developer. By definition we take $(\Delta)_n(\alpha+\beta+\cdots+\gamma) \equiv (\alpha+\beta+\cdots+\gamma)' = \alpha'+\beta'+\cdots+\gamma'$, and this is equal to $(\alpha)_n+(\beta)_n+\cdots+(\gamma)_n$, with a similar definition when some of the signs are negative; $(\Delta)_n(\alpha\beta\cdots\gamma) \equiv (\alpha\beta\cdots\gamma)' = (\alpha, \beta, \cdots, \gamma)_n$, giving the development for a product of elements; and for the development of a quotient

$$(\Delta)_n\left(\frac{\alpha\beta\cdots\gamma}{\varkappa\lambda\cdots\sigma}\right) \equiv \left(\frac{\alpha\beta\cdots\gamma}{\varkappa\lambda\cdots\sigma}\right)' = \left(\frac{\alpha, \beta, \cdots, \gamma}{\varkappa, \lambda, \cdots, \sigma}\right)_n.$$

There can be no confusion if we simplify the writing by putting

$$(\alpha\beta\cdots\gamma)' \equiv \alpha'\beta'\cdots\gamma', \qquad \left(\frac{\alpha\beta\cdots\gamma}{\varkappa\lambda\cdots\sigma}\right)' = \frac{\alpha'\beta'\cdots\gamma'}{\varkappa'\lambda'\cdots\sigma'},$$

so that the multiplications indicated in the first, $\alpha'\beta'\cdots\gamma'$, are purely symbolic, and likewise for the multiplications $\alpha'\beta'\cdots\gamma'$, $\varkappa'\lambda'\cdots\sigma'$ and the division $\alpha'\beta'\cdots\gamma'/\varkappa'\lambda'\cdots\sigma'$ in the second.

THEOREM II'''. *The rational relation* $R(\alpha, \beta, \cdots, \gamma) = 0$ *implies* $R(\alpha', \beta', \cdots, \gamma') = 0$.

This is an immediate consequence of the definitions and Theorems II, II''. A rational relation may in general be written in several distinct forms, say $\alpha\beta+\beta\gamma = \delta$, or $\alpha+\gamma = \beta/\delta$. By the theorem these imply $\alpha'\beta'+\beta'\gamma' = \delta'$, $\alpha'+\gamma' = \delta'/\beta'$, which are $(\alpha\beta)'+(\beta\gamma)' = \delta'$, $\alpha'+\gamma' = (\delta/\beta)'$. The accented relation is of the kind in the theorem for algebra C, and it has been obtained by development $(\Delta)_n$ from the unaccented relation between the elements $\alpha, \beta, \cdots, \gamma$. The extension to irrational relations follows as in Theorem II''.

By the definition of $(\Delta)_n$ combined with the specific properties of a C developer with relation to sums, etc., it follows that $\Delta_m(\Delta_n(\alpha\beta))$ must be identical with $\Delta_{m+n}(\alpha\beta)$, with the like for sums, differences and quotients, if the definition and specified properties are to be consistent. It therefore remains to show that for at least one class of elements $\alpha, \beta, \cdots, \gamma$ and one developer $(\Delta)_n$ these conditions are satisfied. The following is an important example.

Let the $\alpha, \beta, \cdots, \gamma$ denote single valued functions of x such that all their derivatives exist and are finite and single valued when $x = 0$. (Any number other than 0 might be chosen, but the statement is simplest in this case.) Let Δ_n be the operation of taking the nth derivative of a given function with respect to x, replacing x by 0 in the result, and finally multiplying this by $x^n/n!$ Then so far as sums and differences are concerned it is obvious that $(\Delta)_n$ satisfies the conditions in question. The like follows at once for products from Leibniz's theorem for the successive derivatives of a product. Hence the conditions are satisfied also for a quotient.

We have $(\Delta)_n \alpha = (\alpha)_n$; and if $(\alpha)_n \sim \sum \alpha_n \equiv \sum \Delta_n \alpha$, we call the series $\sum \Delta_n \alpha$ the $(\Delta)_n$ development of α. In the above special example the $(\Delta)_n$ development is the Maclaurin series. The $(\Delta)_n$ development in every case is unique and significant, and from the foregoing theorems such series can be formally manipulated in any system.

3. **Algebra** D. By the remarks on notation in § 1 (7), it is unnecessary to repeat the definition and properties of addition and subtraction for sequences written in the $\{a\}_n$ notation. For uniformity with what follows these operations in this case are called D addition and subtraction. The D product of the sequences $\{a\}_n, \{b\}_n, \cdots, \{k\}_n$ is the sequence $\{a, b, \cdots, k\}_n$ whose nth element ${}_n\{a, b, \cdots, k\}$ is defined by

$$ {}_n\{a, b, \cdots, k\} = \sum a_\alpha b_\beta \cdots k_\varkappa, \qquad \alpha\beta\cdots\varkappa = n, $$

$$ 1 \leqq \alpha, \beta, \cdots, \varkappa \leqq n, $$

the notation indicating that the summation extends to all possible sets of positive integers $\alpha, \beta, \cdots, \varkappa$ whose product is n, the number of integers in each set being equal to the number of sequences whose product is taken. Clearly $\{a, b, \cdots, k\}_n$ is invariant under all permutations of $a, b, \cdots, k$. Hence D multiplication is commutative. To see that it is also associative let $\{a\}_n, \{b\}_n, \cdots, \{k\}_n$ be distributed in any way into sets such that each sequence occurs once and once only in all the sets; form the D product for all the sequences in each of the sets, and finally take the D product of these products. Then evidently the result is the D product of all the sequences. Similarly it is seen at once that with respect to D addition and subtraction D multiplication is distributive. The unit with respect to this multiplication is the sequence $\{v\}_n, v_1 = \epsilon, v_n = 0\ (n > 1)$.

The sequence $\{a'\}_n$ satisfying $\{a, a'\}_n = \{v\}_n$, where $\{a\}_n$ is not exceptional, is called the D reciprocal of $\{a\}_n$ and is written $\{v/a\}_n$. To simplify the printing put for a moment $(i, j) = 0$ or $a_{i/j}$ according as i/j is not or is an integer > 0. Then proceding as in C we find

$$a'_n=(-1)^{n-1}a_1^{-n}\begin{vmatrix} (n,n-1) & (n,n-2) & \cdots & (n,2) & (n,1) \\ (n-1,n-1) & (n-1,n-2) & \cdots & (n-1,2) & (n-1,1) \\ 0 & (n-2,n-2) & \cdots & (n-2,2) & (n-2,1) \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots & \cdots\cdots \\ 0 & 0 & \cdots & (3,2) & (3,1) \\ 0 & 0 & \cdots & (2,2) & (2,1) \end{vmatrix},$$

which with $a'_1=\epsilon/a_1$ completely defines $\{v/a\}_n$. Continuing, with the necessary and obvious modifications, as in C we infer that the first of the following relations implies the second, none of $\{a\}_n, \{b\}_n, \cdots, \{c\}_n$ being exceptional:

$$\{a,b,\cdots,c,d,e,\cdots,f\}_n=\{k,l,\cdots,s\}_n,$$

$$\{d,e,\cdots,f\}_n=\{k,l,\cdots,s,v/a,v/b,\cdots,v/c\}_n,$$

or, as we shall write it,

$$\{d,e,\cdots,f\}_n=\left\{\frac{k,l,\cdots,s}{a,b,\cdots,c}\right\}_n.$$

THEOREMS III, IV, IV″. *Replace C by D in Theorems* I, II, II″.

The connection with Dirichlet series is obtained thus. For all integral values >0 of n write $a_n=a_nn^t$, $\beta_n=b_nn^t,\cdots,\gamma_n=c_nn^t$, and put $\{\alpha\}_n\infty\sum a_nn^t$, $\{\beta\}_n\infty\sum b_nn^t,\cdots$, $\{\gamma\}_n\infty\sum c_nn^t$. Then for a definite choice of the signs,

$$\{\pm\alpha\pm\beta\pm\cdots\pm\gamma\}_n\infty\sum{}_n(\pm a_n\pm b_n\pm\cdots\pm c_n)n^t;$$

$${}_n\{\alpha,\beta,\cdots,\gamma\}=\{a,b,\cdots,c\}n^t,$$

and to find the series associated with $\{\alpha,\beta,\cdots,\gamma\}_n$ we multiply together formally the associated series of $\{\alpha\}_n, \{\beta\}_n,\cdots,\{\gamma\}_n$ and rearrange the result in the form $\sum k_nn^t$:

$$\{\alpha,\beta,\cdots,\gamma\}_n\infty\sum a_nn^t\times\sum b_nn^t\times\cdots\times\sum c_nn^t=\sum k_nn^t.$$

The further development parallels that for algebra C upon replacing the $(a)_n$ notation by the $\{\alpha\}_n$ and referring the discussion to coefficients of n^t instead of t^n as before.

Corresponding to the alternative for C we have for D the algebra in which the units ϵ_n are commutative, with the law of combination $\epsilon_m\epsilon_n=\epsilon_{m+n}$. This algebra is the direct product of an infinite number of algebras C, generated by a modulus and units $e_1, e_2,\cdots$ respectively. The correspondence between the algebras is as follows. Let $n=\alpha^a\beta^b\cdots$ be the resolution of n into its prime factors, and let α be the λth prime, β the μth prime, etc. Then $\epsilon_n=\epsilon_\lambda^a\epsilon_\mu^b\cdots$.

II. ALGEBRA E

4. By the proper change in notation any sequence may be written in the form $a_2, a_3, a_4, \cdots, a_n, \cdots$, and if we put $a_1 = \varepsilon$ (§ 1(3)) *whatever the sequence*, we consider throughout this part $\{a\}_n$, that is, $a_1 (= \varepsilon), a_2, a_3, \cdots$, instead of the original sequence $a_2, a_3, \cdots$. This inclusion of ε as the first element in every sequence is merely a device to shorten the statements of processes and results in what follows. We now make a radical change in notation and elaborate the concepts upon which E is constructed.

(10) Having written our initial sequence as above prescribed in the form $\{a\}_n$ with first element ε, we next put $a_1 = x_1, a_2 = x_2, a_3 = x_3, a_4 = x_5, a_5 = x_7, \cdots, a_n = x_p, \cdots$, where p is the nth prime number $\geqq 1$. We thus obtain $\{a\}_n$ in the *normal script* $x_1, x_2, x_3, x_5, x_7, x_{11}, x_{13}, x_{17}, \cdots, x_p, \cdots$, which is written $'\{x\}_p$ and is called a *primary sequence*. The first element x_1 of $'\{x\}_p$ is ε.

This again is nothing more than an artifice by which the algebra is greatly simplified in expression. In the extension of E processes to multiple sequences (not discussed here) the inherent complexity of the generalized concepts is such that a simplified notation is imperative. The extension of the above device is immediate.

(11) The aggregate X of all formal products $x_\alpha^a x_\beta^b \cdots x_\gamma^c$ of positive integral or zero powers $x_\alpha^a, x_\beta^b, \cdots, x_\gamma^c$ of distinct elements $x_\alpha, x_\beta, \cdots, x_\gamma$ of $'\{x\}_p$ is not a sequence until a law is assigned by which the several products are ordered. We adopt first the convention that each product of this type in which all of the exponents $a, b, \cdots, c$ are zero is identical with $x_1 (\equiv \varepsilon)$. Since $x_1 x_n = x_n$ it follows that $'\{x\}_p$ is contained in X, that is, every element of $'\{x\}_p$ is an element of X. If now $n = \alpha^a \beta^b \cdots \gamma^c$, where $a, b, \cdots, c$ are all different from zero, is the resolution of n into its prime factors > 1, we define x_n by $x_n = x_\alpha^a x_\beta^b \cdots x_\gamma^c$ and form the sequence $\{x\}'_n \equiv x_1, x_2, x_3, x_4, \cdots, x_n, \cdots$, called the *first derivative* of the primary $'\{x\}_p$.

Each product is now placed; for example $x_2^5 x_7^2 x_{11}^3 = x_{20870008}$. Conversely the resolution of each element of $\{x\}'_n$ into a product of powers of distinct elements of $'\{x\}_p$ is uniquely known; thus $x_{7605} = x_3^2 x_5 x_{13}^2$. If from $\{x\}'_n$ we delete all elements whose suffixes are composite there remains $'\{x\}_p$.

Now obviously $\{x\}'_n$ may be written in normal script as a primary sequence, say $''\{x\}_p$, and we can take its first derivative $''\{x\}'_n$, or more briefly $\{x\}''_n$, which is called the second derivative of $\{x\}'_p$. Continuing thus we define the rth derivative for r any integer $\geqq 1$. For simplicity we consider here only the case $r = 1$.

(12) Let $a, b, \cdots, c, \cdots$ denote an infinity of integers each of which may vary from zero to positive infinity, and let $\alpha, \beta, \cdots, \gamma, \cdots$ denote the primes > 1 in any order. Denote by any one of the infinity of symbols $f_0(x_\delta)$, where

$\delta > 1$ is prime, the unit with respect to multiplication in the abstract annulus (§ 1 (4)) whose base is the aggregate of elements $f_a(x_a), f_b(x_\beta), \cdots, f_c(x_\gamma), \cdots$ where $a, b, \cdots, c, \cdots$ and $\alpha, \beta, \cdots, \gamma, \cdots$ take all values consistent with their definitions; and let each element of the annulus have a unique significance in terms of elements of $'\{x\}_p$. Then for specific a and α, $f_a(x_a)$ is called a *primary function* of x_a, and similarly for $f_b(x_\beta)$ and $x_\beta, \cdots, f_c(x_\gamma)$ and $x_\gamma, \cdots$; and these primary functions are *associated with* the elements $x_a^a, x_\beta^b, \cdots, x_\gamma^c, \cdots$ of $\{x\}'_n$ respectively. From the definition of the unit, $f_0(x_a) f_b(x_\beta) = f_b(x_\beta)$, $f_0(x_a) f_0(x_\beta) = f_0(x_a) = f_0(x_\beta)$, and so on.

(13) The notation being as in (12), so that $\alpha > 1$ is prime, the sequence $f_0(x_a)$, $f_1(x_a), f_2(x_a), \cdots, f_n(x_a), \cdots$, which we shall write $(f(x_a))_n$, is called a *primary functional sequence.*

(14) We next order the products contained in the annulus of (12) into a sequence. Let $n = \alpha^a \beta^b \cdots \gamma^c$ be the resolution of $n > 1$, and recall that by (11) the element x_n of $\{x\}'_n$ is $x_a^a x_\beta^b \cdots x_\gamma^c$. With this element we associate the product $f_a(x_a) f_b(x_\beta) \cdots f_c(x_\gamma)$ and denote it by either of $f_x(n)$ or $f_n(x)$. If m is not prime $f_n(x_m)$ is meaningless; also $f_1(x)$ is not yet defined. As suggested by $1 = \alpha^0 = \beta^0 = \cdots = \alpha^0 \beta^0 =$ etc., we now identify $f_1(x)$ with the unit $f_0(x_a)$, $\equiv f_0(x_\beta) \cdots \equiv f_0(x_\gamma)$, etc., defined in (12). When n is given, the place of x_n in $\{x\}'_n$ is known, and likewise for $f_n(x)$ in the sequence $f_1(x), f_2(x), f_3(x), \cdots, f_n(x), \cdots$, which we shall denote by $\{f(x)\}_n$ and call a *derived functional sequence.* Obviously each product contained in the annulus occurs once and once only in $\{f(x)\}_n$, which also contains all primary functional sequences $(f(x_p))_n$, p running through all primes $p > 1$.

The x in $f_x(n)$ or $f_n(x)$ indicates only that the arguments of the several $f_a, f_b, \cdots, f_c$ in the formal product which is denoted by either of these symbols are elements of the sequence $\{x\}'_n$, and x itself is *not* one of these arguments, for it is not an element of the sequence. If the sequence were $\{y\}'_n$ our symbol would be written $f_n(y)$ or $f_y(n)$. The x in $f_n(x)$ is thus purely umbral, and has not the usual significance that $f_n(x)$ is a function of x. But on the other hand the element x_n being uniquely determined when n is assigned, $f_x(n)$ is in the ordinary sense a function of the positive non-zero integer n.

When several derived functional sequences $\{f(x)\}_n, \{g(x)\}_n, \cdots$ are being considered together we postulate that $f_1(x), g_1(x), \cdots$ are identical and that (as in § 1 (1)) the formal sums, differences and products constructed from elements of all the sequences are uniquely significant.

As an example of the notation, $f_{360}(x) = f_3(x_2) f_2(x_3) f_1(x_5)$, since $360 = 2^3 3^2 5$ and hence $x_{360} = x_2^3 x_3^2 x_5$.

5. ***E* composition.** This process, which is fundamental for E, has the formal properties of multiplication and is radically distinct from any in C, D, although, as will appear, it combines both C and D multiplication. The derived functional sequence $\{f(x)\}_n$ is uniquely determined (§ 4 (14)) by the aggregate

of primary functional sequences $(f(x_p))_n$ where p runs through all primes $p>1$. Conversely, *all* of the sequences in this aggregate are necessary for the determination of $\{f(x)\}_n$, and the aggregate defines one and only one derived functional sequence. Let us express this by saying that $\{f(x)\}_n$ is the E composite $[(f(x_p))_n]$ of all the sequences in the aggregate of primary functional sequences $(f(x_p))_n$, and let us write this symbolically as follows:

$$[(f(x_p))_n] = \{f(x)\}_n,$$

which may be read: the E composite of all the sequences in the aggregate $(f(x_p))_n$ where p runs through all primes >1 is the sequence $\{f(x)\}_n$, and this sequence is obtained from the aggregate by E composition.

(15) The E composites $[(f(x_p))_n]$, $[(g(x_p))_n]$ are defined to be equal, $[(f(x_p))_n] = [(g(x_p))_n]$, when and only when $\{f(x)\}_n = \{g(x)\}_n$.

Theorems for E operations are thus restatements in terms of E composites of theorems in a D algebra in which the elements are derived functional sequences, and they follow at once from Theorems III, IV, IV″, upon translating them into their equivalents in terms of derived functional sequences. The discussion will be more easily followed from the series equivalents of E composition which we consider next.

6. To obtain the formal equivalents of E operations we require the parameters t_n which are defined on replacing x in § 4(10), (11) by t. Thus $n = \alpha^a \beta^b \cdots \gamma^c$ being as before the prime factor resolution of n, the element t_n of $\{t\}'_n$ is $t_\alpha^a t_\beta^b \cdots t_\gamma^c$, $t_\alpha^0 = t_1 = \varepsilon$ (α any prime), $t_1 t_n = t_n$; and if $n = \alpha^a$, $t_n = t_\alpha^a$. With respect to the elements of derived functional sequences $\{f(x)\}_n$, $\{g(x)\}_n, \cdots$, t_n has all the properties of formal multiplication so that $t_n f_m(x) = f_m(x)\, t_n$, and if $t_n f_m(x) = t_n g_m(x)$, then $f_m(x) = g_m(x)$.

Write for a moment $t_p^n f_n(x_p) = f'_n(x_p)$. Then $\sum_0 f'_n(x_p) \sim (f'(x_p))_n \sim \sum_0 t_p^n f_n(x_p)$. From the definition of E composition and (15) the sequence $\{f'(x)\}_n$ is uniquely determined by the aggregate of sequences $(f'(x_p))_n$, where p runs through all primes >1. Hence since $\{f'(x)\}_n \sim \sum_1 f'_n(x)$, the series $\sum_1 t_n f_n(x)$ is uniquely determined by the aggregate of sequences $(f'(x_p))_n$, and therefore by the aggregate of associated series $\sum_0 t_p^n f_n(x_p)$. In other words when the general term in each series of the aggregate is known, so also is the general term $t_n f_n(x)$, and evidently this term can be found by multiplying together formally all the series in the aggregate, or as we shall write it symbolically

$$\prod_p \left[\sum_0 t_p^n f_n(x_p)\right] = \sum_1 t_n f_n(x),$$

for the coefficient of $t_n = t_\alpha^a t_\beta^b \cdots t_\gamma^c$ in the formal product is $f_a(x_\alpha) f_b(x_\beta) \cdots f_c(x_\gamma) = f_n(x)$. This is the formal equivalent of E composition.

7. **Generators.** These play a part in E analogous to that of logarithms in common arithmetic. In the typical factor of the above formal product write $t_p \equiv t$, $x_p \equiv \xi$, so that it becomes $F(t, \xi) \equiv \sum_0 t^n f_n(\xi)$, which is to be regarded as a function of two independent variables t, ξ. We define $F(t, \xi)$ to be the *generator* of $f_n(x)$, and write

$$F(t, \xi) \cdot \Gamma \cdot f_n(x),$$

which is to be read: $F(t, \xi)$ is the generator of $f_n(x)$.

To write down the $f_n(x)$ generated by $F(t, \xi) \equiv \sum_0 t^n f_n(\xi)$ we first set $f_n(x) = f_a(x_\alpha) f_b(x_\beta) \cdots f_c(x_\gamma)$, where $n = \alpha^a \beta^b \cdots \gamma^c$ is the prime factor resolution of n, and then in this substitute for $f_a(x_\alpha), f_b(x_\beta), \cdots, f_c(x_\gamma)$ as defined by $f_n(x)$ which is given, since $F(t, \xi)$ is assumed known. Thus $f_a(x_\alpha)$ is obtained from $f_n(\xi)$ by putting $n = a$, $\xi = x_\alpha$. E. g., if $f_n(\xi) = \xi^n$, $F(t, \xi) \equiv \sum_0 t^n \xi^n$ generates $x_\alpha^a x_\beta^b \cdots x_\gamma^c = x_n$. Conversely, to write down the generator when the form of $f_n(x)$ is known, $= f_a(x_\alpha) f_b(x_\beta) \cdots f_c(x_\gamma)$, we replace in any factor, say $f_a(x_\alpha)$, the argument x_α by ξ and the suffix by n, getting $f_n(\xi)$, multiply by t^n and sum for $n = 0, 1, 2, \cdots$: thus $\sum_0 t^n f_n(\xi)$, $= F(t, \xi)$.

Either from our definitions or by convention t^0 has with respect to powers of t the multiplicative properties of unity, and similarly for each of the identical symbols $f_0(\xi)$, $g_0(\xi), \cdots$ with respect to each of $f_n(\xi)$, $g_n(\xi), \cdots$. Only these properties will be used in generators, so that each of these symbols may be replaced by 1. A generator therefore is a power series in t whose first term is 1 and in which the coefficients of the several powers of t are functions of ξ (some or all of which may reduce to constants). With each generator is associated (§ 1 (9.1)) a sequence, and these sequences are combined only according to C multiplication and division; with the sequences resulting from these operations are associated series, and clearly each of these series is a generator. Hence, under C, multiplication (and division) of series generators form a group.

Algebra E is concerned with the relations between the $f_n(x)$, $g_n(x), \cdots$ generated by the members of this group. Specific applications of E are determined by special choices of the initial generators (e. g., each $f(\xi)$, $g(\xi), \cdots$ is an algebraic function of ξ), and specific interpretations of the system of elements x_n upon which the $f, g, \cdots$ are constructed (e.g., $x_n = n$, giving ordinary arithmetic). By means of the Maclaurin development after Theorem II''' it may or may not be possible in a given application to reduce all of the generators in the group to finite form. This is the case, however, in one of the most important applications (§ 12), when generators are combined according to the rules for the multiplication of ordinary algebraic fractions in two letters.

Understanding that the arguments are t, ξ we may shorten $F(t, \xi)$ to F, and similarly $f_n(x)$ may be abbreviated to f, so that the generational relation will be written where convenient as $F \cdot \Gamma \cdot f$.

8. The definition (15) and § 5 yield an important consequence. Take the C product, (§ 2), $(f(x_p), g(x_p), \cdots, h(x_p))_n$, of the primary functional sequences $(f(x_p))_n$, $(g(x_p))_n, \cdots, (h(x_p))_n$, noting that the argument x_p, and hence the prime $p > 1$, is the same in all. If for a moment we denote the nth element of this C product by $\varphi_n(x_p), \equiv {}_n(f(x_p), g(x_p), \cdots, h(x_p))$ we have $(\varphi(x_p))_n$ a primary functional sequence, and we can form the E composite $[(\varphi(x_p))_n] \equiv [(f(x_p), g(x_p), \cdots, h(x_p))_n]$. On the other hand we form the D product, (§ 3), $\{f(x), g(x), \cdots, h(x)\}_n$ of the derived functional sequences $\{f(x)\}_n$, $\{g(x)\}_n, \cdots, \{h(x)\}_n$, and have at once the following:

THEOREM V. *The E composite and the D product are identical*:

$$[(f(x_p), g(x_p), \cdots, h(x_p))_n] = \{f(x), g(x), \cdots, h(x)\}_n.$$

The content of this theorem will be evident from its equivalent in generators, given next. From now on the algebra may be developed either from the standpoint of sequences or from that of generators. The latter leading at once to the extremely simple working rules of E, we choose it.

9. **E multiplication.** Consider the r relations

$$F \cdot \Gamma \cdot f, \qquad G \cdot \Gamma \cdot g, \qquad \cdots, \qquad H \cdot \Gamma \cdot h.$$

The product $K \equiv FG \cdots H$ is a generator, say $K \cdot \Gamma \cdot k$. We define k to be the E product $fg \cdots h$ of $f, g, \cdots, h$. Or writing this in full, $k_n(x) \equiv (fg \cdots h)_n(x)$. That is, the generator of the E product $fg \cdots h$ of $f, g, \cdots, h$ is the product of the several generators. This merely defines the symbol $fg \cdots h$ which must now be evaluated. Evidently $k_n(x)$ is the nth term ${}_n\{f(x), g(x), \cdots, h(x)\}$ of the D product of the r sequences $\{f(x)\}_n$, $\{g(x)\}_n, \cdots, \{h(x)\}_n$. Hence the explicit form of $k_n(x)$ is $fg \cdots h \equiv (fg \cdots h)_n(x) \equiv \sum_n f_\delta(x) g_\delta(x) \cdots h_\tau(x)$, where the summation refers to all sets $(\delta, \delta, \cdots, \tau)$ of r integers each > 0 whose product is n.

The detailed proof is the same for any number of factors: let $r = 2$. The C product $K(t, \xi)$ of $F(t, \xi) \equiv \sum_0 t^n f_n(\xi)$ and $G(t, \xi) \equiv \sum_0 t^n g_n(\xi)$ is $\sum_0 t^n k_n(\xi)$, where $k_n(\xi) = \sum_{r=0}^{n} f_r(\xi) g_{n-r}(\xi)$; and by the definition of a generator, $K(t, \xi) \cdot \Gamma \cdot k_n(x)$, where, $n = \alpha^a \beta^b \cdots \gamma^c$ being the prime factor resolution of n, $k_n(x) = k_a(x_\alpha) k_b(x_\beta) \cdots k_c(x_\gamma)$. It is to be shown that $k_n(x) = {}_n\{f(x), g(x)\}$. Let $n = d\delta$ be the resolution of n into any pair of conjugate divisors > 0, and $d = \alpha^{a_1} \beta^{b_1} \cdots \gamma^{c_1}$, $\delta = \alpha^{a-a_1} \beta^{b-b_1} \cdots \gamma^{c-c_1}$ the prime factor resolutions of d, δ.

In these some of the exponents may be zeros. Substituting in $k_n(x)$ the values of $k_a(x_\alpha)$, $k_b(x_\beta), \cdots, k_c(x_\gamma)$ defined by the generator $K(t, \xi)$ we get for $k_n(x)$ the expression

$$\Big[\sum_{a_1=0}^{a} f_{a_1}(x_\alpha)\, g_{a-a_1}(x_\alpha)\Big]\Big[\sum_{b_1=0}^{b} f_{b_1}(x_\beta)\, g_{b-b_1}(x_\beta)\Big]\cdots\Big[\sum_{c_1=0}^{c} f_{c_1}(x_\gamma)\, g_{c-c_1}(x_\gamma)\Big]$$
$$= \sum f_{a_1}(x_\alpha) f_{b_1}(x_\beta)\cdots f_{c_1}(x_\gamma)\, g_{a-a_1}(x_\alpha)\, g_{b-b_1}(x_\beta)\cdots g_{c-c_1}(x_\gamma),$$

the $\sum$ referring to all $a_1, b_1, \cdots, c_1$ defined by $0 \leqq a_1 \leqq a$, $0 \leqq b_1 \leqq b, \cdots, 0 \leqq c_1 \leqq c$. But

$$f_{a_1}(x_\alpha) f_{b_1}(x_\beta)\cdots f_{c_1}(x_\gamma) = f_d(x), \quad g_{a-a_1}(x_\alpha) g_{b-b_1}(x_\beta)\cdots g_{c-c_1}(x_\gamma) = g_\delta(x).$$

Hence $k_n(x) = \sum_n f_d(x) g_\delta(x)$, the sum referring to all pairs of divisors $d, \delta > 0$ of n such that $d\delta = n$. That is, $k_n(x) = {}_n\{f(x), g(x)\}$.

Obviously $fg\cdots h$ is invariant under all permutations of $f, g, \cdots, h$; hence F multiplication is commutative. That it is also associative follows immediately as in D.

(16) The *unit* of E multiplication is $\eta \equiv \eta_n(x) \equiv \eta_x(n)$ defined by $\eta_x(1) = 1$, $\eta_x(n) = 0\ (n > 1)$.

10. **E division.** By definition the E reciprocal f' of f is generated by the reciprocal F' of the generator F of f. In this F' is obtained by the formal division of 1 by F. Since $FF' = 1$, FF' is the generator of η.

With respect to E multiplication and division η has the properties of unity in ordinary algebraic multiplication and division, for $\eta f = f$, so that the factor η may be omitted from any E product, and evidently η is its own reciprocal. Since F is any generator, η has an infinity of generators. By an obvious algebraic analogy, $ff' = \eta$ may be written when convenient in either of the forms $f = \eta/f'$, $f' = \eta/f$.

The E reciprocal of an E product is obviously the E product of the several E factors of the product:

$$\frac{\eta}{fg\cdots h} = \frac{\eta}{f}\frac{\eta}{g}\cdots\frac{\eta}{h}.$$

The E product of $fg\cdots h$ and $\eta/\varphi\psi\cdots\chi$ is written in the form

$$\frac{fg\cdots h}{\varphi\psi\cdots\chi},$$

and clearly such E fractions have all the multiplicative properties of ordinary fractions.

11. **E addition and subtraction.** Since by § 4 (14) (end) the $f_n(x)$, $g_n(x), \cdots$ form an annulus, $\pm f_n(x) \pm g_n(x) \pm \cdots$, for a definite choice of the signs in an element $l_n(x)$ of the annulus. Shortening the notation as before we write, for a definite choice of the signs $l \equiv \pm f \pm g \cdots$, thus defining E addition and subtraction. For the same choice of signs the E product of φ and l is φl, and at once from the definitions we have

$$\varphi l = \pm \varphi f \pm \varphi g \pm \cdots.$$

Since E addition and subtraction are obviously associative these operations have all the formal properties of algebraic addition and subtraction.

In full for $l = f + g$: let $l_n(x) = f_n(x) + g_n(x)$. By the definition of E multiplication, $(\varphi l)_n(x) = \sum_n \varphi_d(x) l_\delta(x)$, the sum extending to all pairs of integers $d, \delta > 0$ such that $d\delta = n$. Hence $(\varphi l)_n(x) = \sum_n \varphi_d(x)[f_\delta(x) + g_\delta(x)] = \sum_n \varphi_d(x) f_\delta(x) + \sum_n \varphi_d(x) g_\delta(x)$ $= (\varphi f)_n(x) + (\varphi g)_n(x)$; viz., $\varphi l = \varphi f + \varphi g$.

THEOREM VI. *The aggregate of all the elements of all derived functional sequences forms a system with respect to the four E rational operations.*

Fractional and negative powers are introduced precisely as in C, and combining these with Theorem VI, we have the complete algebra E.

We note that the product $f_n(x)\, g_n(x) \cdots h_n(x)$ is an element of the above aggregate. It cannot be abbreviated to $fg \cdots h$ in applying E, for this denotes the E product. We accordingly use $|fg \cdots h|$, writing therefore

$$|fg \cdots h|_n(x) \equiv f_n(x)\, g_n(x) \cdots h_n(x).$$

With this we form E products. Thus the E product of $|fg|$ and k is written $|fg|\, k$, and is in full $\sum_n f_d(x)\, g_d(x)\, k_\delta(x)$.

The algebra is used as outlined in the following example.*

12. **Example.** Let

$$F(t, \xi) \equiv 1 + t f_1(\xi) + \cdots + t^n f_n(\xi)$$

be a polynomial in the two independent variables t, ξ of degree $n > 0$ in t, and let all the coefficients of F lie in the domain of rationality ∇. We call F

* Detailed applications of this have been made with numerous special theorems as examples, by Miss E. D. Pepper, Tôhoku Mathematical Journal, vol. 22, Nos. 1, 2. The theorems include many of those mentioned at the end of § 1 and are sufficient in number to show the increase in power and the great saving in space and labor of manipulation gained by these methods. Finally the paper contains applications of E to the multiplicative properties of algebraic numbers, the ideals of any realm, and systems of polynomials to a single or a double modulus. It will therefore be sufficient here to sketch the development in one case.

and $1/F$ primary generators, which are prime or composite according as F is irreducible or reducible in ∇. The aggregate P of all primary generators obtained by taking $n = 1, 2, \cdots$, and $f_1(\xi), f_2(\xi), \cdots$ successively equal to all polynomials in one variable ξ with coefficients in ∇ may be arranged in pairs $(F, F'), (G, G'), \cdots$, such that either generator in a pair is the reciprocal of the other. If $F \cdot \Gamma \cdot f$, $F' \cdot \Gamma \cdot f'$, then since $f' = \eta/f$, $f = \eta/f'$, we may take as fundamental either f or f', regarding the other as its reciprocal. We shall choose as fundamental that one of f, f' whose generator is a polynomial, viz., is in finite form. For example, of f, f' generated by $1 - t\xi$ and $\sum_0 t^n \xi^n$ respectively, f is fundamental. The generator F of a fundamental f is called fundamental, and f is prime or composite (in ∇) according as F is prime or composite.

If in the E product $fg \cdots h$ the r factors $f, g, \cdots, h$ are identical, it is written f^r, and by convention $f^0 = \eta$. Obviously if $F \cdot \Gamma \cdot f$, then $F^r \cdot \Gamma \cdot f^r$ and $F'^r \cdot \Gamma \cdot f'$ where $F' = 1/F, f' = \eta/f$. Any fundamental generator may be resolved uniquely in ∇ into a product of powers of distinct prime fundamental generators. Let $F = G^a H^b \cdots K^c$ be the prime resolution of F, and let $F \cdot \Gamma \cdot f$, $G \cdot \Gamma \cdot g$, etc. Then $f = g^a h^b \cdots k^c$ is called the prime resolution of f (in ∇), and similarly for $f' = \eta/f = \eta/g^a h^b \cdots k^c$. These resolutions are unique.

Next consider the aggregate Π of all generators formed from the products $1, 2, 3, \cdots$ at a time of all the generators in P. Any generator in Π is of the form F/Φ, where F, Φ are primary and fundamental, so that F, Φ are in finite form and are not resolved into their factors in ∇. If F, Φ are relatively prime in ∇, F/Φ is called reduced, and likewise for f/φ where $F/\Phi \cdot \Gamma \cdot f/\varphi$. If in this case $f = g^a h^b \cdots k^c$, $\varphi = \lambda^m \mu^n \cdots \varrho^s$ are the prime resolutions of f, φ, we call $g^a h^b \cdots k^c/\lambda^m \mu^n \cdots \varrho^s$ the prime resolution of f/φ. If $F = TF_1$, $\Phi = T\Phi_1$ where F_1, Φ_1 are relatively prime in ∇, $F/\Phi = F_1/\Phi_1$ and the latter, being reduced, yields the prime resolution of f/φ. Or the same result can be reached by cancellation of powers of $g, h, \cdots$ against like powers of identical $\lambda, \mu, \cdots$ in f/φ after the prime resolutions of f, φ. From the manner in which it has been obtained the prime resolution of f/φ is unique.

The aggregate of all E quotients f/φ, including all cases in which either f or φ is the unit η, is denoted by A; the generators of these quotients are those in Π.

To investigate the relation between given members of A we first find their prime resolutions and then note the multiplicative relations between the results to obtain identities, performing multiplications and divisions as in arithmetic. Two problems arise which are complementary to one another. First, suppose we are given the generator, which may be of one of the forms

F, $1/F$, F/G, where F, G are fundamental and primary; it is required to find the corresponding f, η/f, f/g. The most general form of F is

$$F(t, \xi) \equiv 1 + t^{a_1} f_{a_1}(\xi) + \cdots + t^{a_s} f_{a_s}(\xi),$$

where none of the polynomials $f_j(\xi)$ with coefficients in ∇ is identically zero, and the a's are integers > 0. There is no difficulty in writing down the $f_n(x)$ generated by this (cf. § 7), but as the expression is complicated when $s > 1$, we omit it. In the forms $1/F$, F/G we first perform the formal divisions and then proceed as in § 7. Next, if we are given the $f_n(x)$ we find its generator by § 7. If this generator is in A, it is a matter of algebra to reduce it to finite form and find its resolution in ∇ into prime factors, whence, noting the functions generated by the several prime generators, we get the prime resolution of f. The identities between the $f, g, \cdots$ may be written down either from the generators or the prime resolutions. Each such identity gives a result of the kind mentioned in the introduction (a simple example is given at the end of the paper). The same identity may be read in many ways; thus $f^2 g^2 h = fgh \cdot gh = f^2 \cdot g^2 \cdot h$, etc., where in $fgh \cdot gh$ we take first the E products fgh, gh and then take the E product of the results. Each such reading gives a theorem.

To get theorems relating to the natural numbers take $x_p = p$ (p any prime), and hence $x_n = n$ (n any integer > 0). This is called the numerical case. It refers to the arithmetical functions built up from the concept of divisibility and the unique factorization law, sometimes called numerical functions.

The aggregate H is characterized by two essential features: (1) each generator in H is expressible in finite form as a rational algebraic function of two variables; (2) the coefficients of each generator are in ∇. An examination of Dickson's *History* (loc. cit.) will show that all of the numerical functions in the literature have simple generators satisfying (1), and that the majority satisfy also (2). The only exceptions are those in which occur, as coefficients of the several powers of t, instead of algebraic functions of ξ, certain simple transcendental functions, such as $(-1)^{\xi}$, or those expressing a quadratic character. Moreover with one exception $\nabla \equiv R(1)$, the rational domain. In the exception $\nabla = R(i\sqrt{3})$.

Even trivial algebraic identities yield results of interest. Thus it is readily seen that in the numerical case the equivalent of

$$\frac{1}{1-t} \cdot \frac{1-t}{1-t\xi} = \frac{1}{1-t\xi}$$

is the totient theorem cited in § 1. Of this kind of theorem some of Gegenbauer's are as complicated as any in the literature.* For example, if $J_\nu(n)$ is Jordan's generalization of Euler's

* Cf. Dickson, *History*, vol. 1, p. 298 (72).

$\varphi(n)$, $J_\nu(n) =$ the number of different sets of ν equal or distinct positive integers $\leqq n$ whose G. C. D. is prime to n, and $u_k(n) = n^k$, and $\rho_{k,s}(n) =$ the sum of the kth powers of those divisors d_s of n whose conjugate divisors are exact sth powers, Gegenbauer gives

$$\sum_n J_\nu(d)\, d^k \rho_{k,s}\left(\frac{n}{d}\right) = \rho_{k+\nu,s}(n),$$

which in E notation is $|J_\nu u_k|\rho_{k,s} = \rho_{k+\nu,s}$. From the above definitions it is easily seen that

$$\frac{1-\xi^k t}{1-\xi^{k+\nu} t} \cdot \Gamma \cdot |J_\nu u_k|, \qquad \frac{1}{(1-\xi^k t)(1-t^s)} \cdot \Gamma \cdot \rho_{k,s}.$$

Hence Gegenbauer's result is the translation of the identity

$$\frac{1-\xi^k t}{1-\xi^{k+\nu} t} \cdot \frac{1}{(1-\xi^k t)(1-t^s)} = \frac{1}{(1-\xi^{k+\nu} t)(1-t^s)}.$$

Noting that $\lambda(n) \equiv +1$ or -1 according as the total number of prime divisors of n is even or odd is generated by $1/(1+t)$, and that Möbius's $\mu(n)$ is generated by $(1-t)$, also $|\mu^2|$ by $(1+t)$, and that the generator of $\sigma_k(n) \equiv$ the sum of the kth powers of all the divisors of n is $1/(1-\xi^k t)$, also that $1/(1-t^s)$ generates $\tau_s(n) \equiv 1$ or 0 according as n is or is not the sth power of an integer > 0, and that $1-\xi^k t$ generates $|\mu u_k|$, we see that the same simple identity yields several more results upon rearrangement of its factors, etc., either for special values of k, ν, s or for general.

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